# A high-order discontinuous Galerkin approach to the elasto-acoustic problem

Francesco Bonaldi

joint work with P. F. Antonietti and I. Mazzieri

MOX, Dipartimento di Matematica

Politecnico di Milano

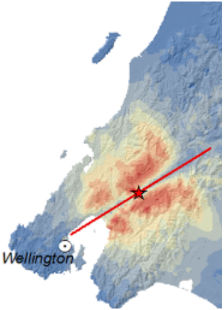
Journées Jeunes EDPistes, Nancy

22 mars 2018

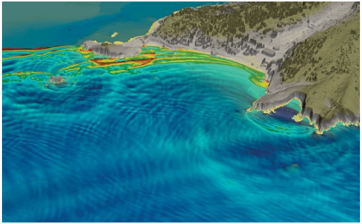
MODELLISTICA E CALCOLO SCIENTIFICO

MODELING AND SCIENTIFIC COMPUTING

# Motivations



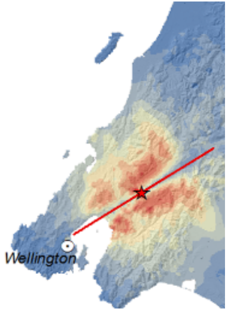

http://speed.mox.polimi.it/



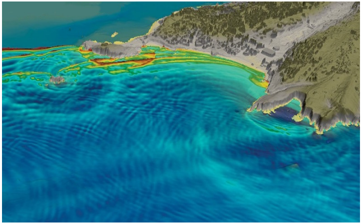
https://www.ictsmarhis.com/

- Simulation of earthquake scenarios near coastal environments
- Coupling of elastic and acoustic wave propagation

# Motivations




http://speed.mox.polimi.it/



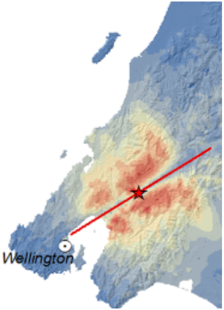
https://www.ictsmarhis.com/

- Simulation of earthquake scenarios near coastal environments
- Coupling of elastic and acoustic wave propagation

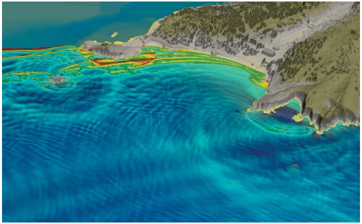
**Requirements on the numerical scheme**

- Flexibility
- Accuracy
- Efficiency

# Motivations




http://speed.mox.polimi.it/



https://www.ictsmarhis.com/

- Simulation of earthquake scenarios near coastal environments
- Coupling of elastic and acoustic wave propagation

**Requirements on the numerical scheme**

- Flexibility
- Accuracy
- Efficiency

**Goal**

- Numerical treatment based on polyhedral meshes
- The dG method supports high-order polynomials on such meshes

# State of the art

## Minimal bibliography

- [Komatitsch *et al.*, 2000]: Spectral Elements
- [Fischer and Gaul, 2005]: FEM–BEM coupling, Lagrange multipliers
- [Flemisch *et al.*, 2006]: classical FEM on two independent meshes
- [Brunner *et al.*, 2009]: FEM–BEM comparison
- [Barucq *et al.*, 2014]: analytical study
- [Barucq *et al.*, 2014]: dG on simplices, curved edges on interface
- [Péron, 2014]: asymptotic study
- [De Basabe and Sen, 2015]: Spectral Elements and Finite Differences
- [Mönköla, 2016]: Spectral Elements, different formulations

# State of the art

## Minimal bibliography

- [Komatitsch *et al.*, 2000]: Spectral Elements
- [Fischer and Gaul, 2005]: FEM–BEM coupling, Lagrange multipliers
- [Flemisch *et al.*, 2006]: classical FEM on two independent meshes
- [Brunner *et al.*, 2009]: FEM–BEM comparison
- [Barucq *et al.*, 2014]: analytical study
- [Barucq *et al.*, 2014]: dG on simplices, curved edges on interface
- [Péron, 2014]: asymptotic study
- [De Basabe and Sen, 2015]: Spectral Elements and Finite Differences
- [Mönköla, 2016]: Spectral Elements, different formulations

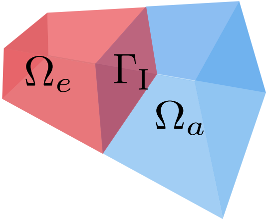
## Our contribution

- **Well-posedness** of the coupled problem in the **continuous setting**
- **Detailed analysis** of a dG scheme on **general polytopic meshes**

# The elasto-acoustic problem

**Governing equations**

$$
\begin{cases}
\rho_e \ddot{\boldsymbol{u}} + 2\rho_e \zeta \dot{\boldsymbol{u}} + \rho_e \zeta^2 \boldsymbol{u} - \mathbf{div}\, \boldsymbol{\sigma}(\boldsymbol{u}) = \boldsymbol{f}_e & \text{in } \Omega_e \times (0,T], \\
\boldsymbol{\sigma}(\boldsymbol{u}) - \mathbb{C}\boldsymbol{\varepsilon}(\boldsymbol{u}) = \mathbf{0} & \text{in } \Omega_e \times (0,T], \\
\boldsymbol{u} = \mathbf{0} & \text{on } \Gamma_{eD} \times (0,T], \\
\boxed{\boldsymbol{\sigma}(\boldsymbol{u})\boldsymbol{n}_e = -\rho_a \dot{\varphi}\, \boldsymbol{n}_e} & \text{on } \Gamma_{\mathrm{I}} \times (0,T], \\
\boldsymbol{u}(0) = \boldsymbol{u}_0, \quad \dot{\boldsymbol{u}}(0) = \boldsymbol{u}_1 & \text{in } \Omega_e, \\
 & \\
c^{-2}\ddot{\varphi} - \triangle \varphi = f_a & \text{in } \Omega_a \times (0,T], \\
\varphi = 0 & \text{on } \Gamma_{aD} \times (0,T], \\
\boxed{\partial \varphi / \partial \boldsymbol{n}_a = -\dot{\boldsymbol{u}} \cdot \boldsymbol{n}_a} & \text{on } \Gamma_{\mathrm{I}} \times (0,T], \\
\varphi(0) = \varphi_0, \quad \dot{\varphi}(0) = \varphi_1 & \text{in } \Omega_a
\end{cases}
$$




- The fluid exerts a pressure on the solid at the interface
- The normal component of the velocity is continuous at the interface

# Well-posedness

**Theorem (Existence and uniqueness)**

Under suitable regularity hypotheses on initial data and source terms, there is a **unique strong solution** s.t.

$$\boldsymbol{u} \in C^2([0,T]; \boldsymbol{L}^2(\Omega_e)) \cap C^1([0,T]; \boldsymbol{H}^1_D(\Omega_e)) \cap C^0([0,T]; \boldsymbol{H}^{\triangle}_{\mathbb{C}}(\Omega_e) \cap \boldsymbol{H}^1_D(\Omega_e)),$$
$$\varphi \in C^2([0,T]; L^2(\Omega_a)) \cap C^1([0,T]; H^1_D(\Omega_a)) \cap C^0([0,T]; H^{\triangle}(\Omega_a) \cap H^1_D(\Omega_a)),$$

$$\boldsymbol{H}^{\triangle}_{\mathbb{C}}(\Omega_e) \coloneqq \{\boldsymbol{v} \in \boldsymbol{L}^2(\Omega_e) : \mathbf{div}\, \mathbb{C}\boldsymbol{\varepsilon}(\boldsymbol{v}) \in \boldsymbol{L}^2(\Omega_e)\},$$
$$H^{\triangle}(\Omega_a) \coloneqq \{v \in L^2(\Omega_a) : \triangle v \in L^2(\Omega_a)\}$$

# Well-posedness

**Theorem (Existence and uniqueness)**

Under suitable regularity hypotheses on initial data and source terms, there is a **unique strong solution** s.t.

$$\boldsymbol{u} \in C^2([0,T];\boldsymbol{L}^2(\Omega_e)) \cap C^1([0,T];\boldsymbol{H}^1_D(\Omega_e)) \cap C^0([0,T];\boldsymbol{H}^{\triangle}_{\mathbb{C}}(\Omega_e) \cap \boldsymbol{H}^1_D(\Omega_e)),$$
$$\varphi \in C^2([0,T];L^2(\Omega_a)) \cap C^1([0,T];H^1_D(\Omega_a)) \cap C^0([0,T];H^{\triangle}(\Omega_a) \cap H^1_D(\Omega_a)),$$

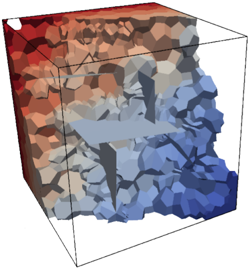
$$\boldsymbol{H}^{\triangle}_{\mathbb{C}}(\Omega_e) := \{\boldsymbol{v} \in \boldsymbol{L}^2(\Omega_e) : \mathbf{div}\, \mathbb{C}\boldsymbol{\varepsilon}(\boldsymbol{v}) \in \boldsymbol{L}^2(\Omega_e)\},$$
$$H^{\triangle}(\Omega_a) := \{v \in L^2(\Omega_a) : \triangle v \in L^2(\Omega_a)\}$$

**Proof**

Apply **Hille–Yosida** upon rewriting the system as

$$\frac{\mathrm{d}\mathcal{U}}{\mathrm{d}t}(t) + A\mathcal{U}(t) = \mathcal{F}(t), \quad t \in (0,T],$$
$$\mathcal{U}(0) = \mathcal{U}_0$$

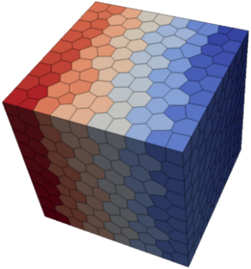
# Meshes and spaces



## Mesh

- Nonconforming **polyhedral** mesh $\mathcal{T}_h = \mathcal{T}_h^e \cup \mathcal{T}_h^a$
- **Arbitrary** number of faces per element
- Possible presence of **degenerating faces**

[Cangiani *et al.*, 2017], [Antonietti *et al.*, 2017]



## Discrete spaces

$$\boldsymbol{V}_h^e = \{\boldsymbol{v}_h \in \boldsymbol{L}^2(\Omega_e) : \boldsymbol{v}_{h|\kappa} \in [\mathscr{P}_{p_{e,\kappa}}(\kappa)]^d \ \forall \kappa \in \mathcal{T}_h^e\},$$

$$V_h^a = \{\psi_h \in L^2(\Omega_a) : \psi_{h|\kappa} \in \mathscr{P}_{p_{a,\kappa}}(\kappa) \ \forall \kappa \in \mathcal{T}_h^a\}$$

# Semi-discrete problem

Find $(\boldsymbol{u}_h, \varphi_h) \in C^2([0,T]; \boldsymbol{V}_h^e) \times C^2([0,T]; V_h^a)$ s.t., for all $(\boldsymbol{v}_h, \psi_h) \in \boldsymbol{V}_h^e \times V_h^a$,

$$\begin{aligned}
&(\rho_e \ddot{\boldsymbol{u}}_h(t), \boldsymbol{v}_h)_{\Omega_e} + (c^{-2}\rho_a \ddot{\varphi}_h(t), \psi_h)_{\Omega_a} \\
&\quad + (2\rho_e \zeta \dot{\boldsymbol{u}}_h(t), \boldsymbol{v}_h)_{\Omega_e} + (\rho_e \zeta^2 \boldsymbol{u}_h(t), \boldsymbol{v}_h)_{\Omega_e} \\
&\quad + \mathcal{A}_h^e(\boldsymbol{u}_h(t), \boldsymbol{v}_h) + \mathcal{A}_h^a(\varphi_h(t), \psi_h) \\
&\quad + \mathcal{I}_h^e(\dot{\varphi}_h(t), \boldsymbol{v}_h) + \mathcal{I}_h^a(\dot{\boldsymbol{u}}_h(t), \psi_h) \\
&\quad = (\boldsymbol{f}_e(t), \boldsymbol{v}_h)_{\Omega_e} + (f_a(t), \psi_h)_{\Omega_a}
\end{aligned}$$

# Semi-discrete problem

Find $(\boldsymbol{u}_h, \varphi_h) \in C^2([0,T]; \boldsymbol{V}_h^e) \times C^2([0,T]; V_h^a)$ s.t.,
for all $(\boldsymbol{v}_h, \psi_h) \in \boldsymbol{V}_h^e \times V_h^a$,

$$
\begin{aligned}
&(\rho_e \ddot{\boldsymbol{u}}_h(t), \boldsymbol{v}_h)_{\Omega_e} + (c^{-2}\rho_a \ddot{\varphi}_h(t), \psi_h)_{\Omega_a} \\
&\quad + (2\rho_e \zeta \dot{\boldsymbol{u}}_h(t), \boldsymbol{v}_h)_{\Omega_e} + (\rho_e \zeta^2 \boldsymbol{u}_h(t), \boldsymbol{v}_h)_{\Omega_e} \\
&\quad + \mathcal{A}_h^e(\boldsymbol{u}_h(t), \boldsymbol{v}_h) + \mathcal{A}_h^a(\varphi_h(t), \psi_h) \\
&\quad + \mathcal{I}_h^e(\dot{\varphi}_h(t), \boldsymbol{v}_h) + \mathcal{I}_h^a(\dot{\boldsymbol{u}}_h(t), \psi_h) \\
&\quad = (\boldsymbol{f}_e(t), \boldsymbol{v}_h)_{\Omega_e} + (f_a(t), \psi_h)_{\Omega_a}
\end{aligned}
$$

## Stability

For sufficiently large stabilization parameters,

$$
\|(\boldsymbol{u}_h(t), \varphi_h(t))\|_{\mathcal{E}} \lesssim \|(\boldsymbol{u}_h(0), \varphi_h(0))\|_{\mathcal{E}} + \int_0^t \left(\|\boldsymbol{f}_e(\tau)\|_{\Omega_e} + \|f_a(\tau)\|_{\Omega_a}\right) \mathrm{d}\tau
$$

# Error estimate

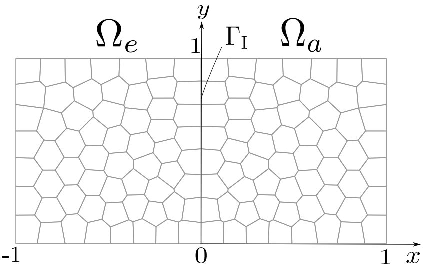
**Energy error estimate**

If $(\boldsymbol{u}, \varphi) \in C^2([0,T]; \boldsymbol{H}^m(\Omega_e)) \times C^2([0,T]; H^n(\Omega_a))$, with $m \geqslant p_e + 1$ and $n \geqslant p_a + 1$, and if stabilization parameters are sufficiently large,

$$\sup_{t\in[0,T]} \|(\boldsymbol{e}_e(t), e_a(t))\|_{\mathcal{E}}^2 \lesssim \frac{h^{2p_e}}{p_e^{2m-3}} \left( \sup_{t\in[0,T]} \left( \|\dot{\boldsymbol{u}}(t)\|_{m,\Omega_e}^2 + \|\boldsymbol{u}(t)\|_{m,\Omega_e}^2 \right) + \int_0^T \left( \|\ddot{\boldsymbol{u}}(t)\|_{m,\Omega_e}^2 + \|\dot{\boldsymbol{u}}(t)\|_{m,\Omega_e}^2 + \|\boldsymbol{u}(t)\|_{m,\Omega_e}^2 \right) \mathrm{d}t \right)$$
$$+ \frac{h^{2p_a}}{p_a^{2n-3}} \left( \sup_{t\in[0,T]} \left( \|\dot{\varphi}(t)\|_{n,\Omega_a}^2 + \|\varphi(t)\|_{n,\Omega_a}^2 \right) + \int_0^T \left( \|\ddot{\varphi}(t)\|_{n,\Omega_a}^2 + \|\dot{\varphi}(t)\|_{n,\Omega_a}^2 + \|\varphi(t)\|_{n,\Omega_a}^2 \right) \mathrm{d}t \right)$$

Optimal in $h$, suboptimal in $p$ by a factor $1/2$

# Numerical example I




## Test case 1

We solve the elasto-acoustic problem on $\Omega_e \cup \Omega_a$, for $T = 1$, $\Delta t = 10^{-4}$, for a homogeneous isotropic elastic material, such that

$$\boldsymbol{u}(x, y; t) = x^2 \cos(\sqrt{2}\pi t) \cos\left(\frac{\pi}{2}x\right) \sin(\pi y)\, \hat{\boldsymbol{u}}, \qquad \hat{\boldsymbol{u}} = (1, 1)$$

$$\varphi(x, y; t) = x^2 \sin(\sqrt{2}\pi t) \sin(\pi x) \sin(\pi y),$$

# Numerical example I

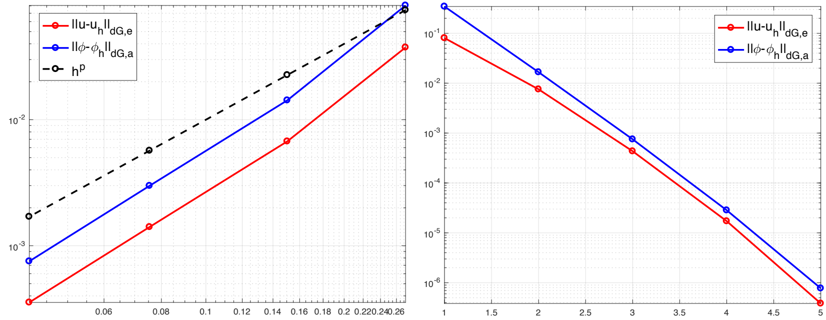
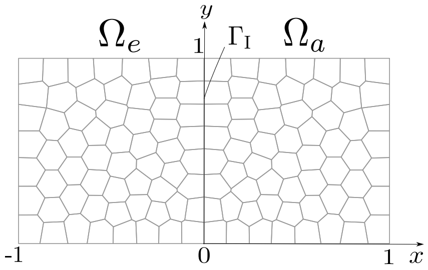



Figure: $\|\boldsymbol{u}-\boldsymbol{u}_h\|_{\mathrm{dG},e}$ and $\|\varphi-\varphi_h\|_{\mathrm{dG},a}$ vs. $h$ (left) and $p$ (right) at $T=1$

# Numerical example II




**Test case 2 [Mönköla, 2016]**

We solve the elasto-acoustic problem on $\Omega_e \cup \Omega_a$, for $T = 0.8$, $\Delta t = 10^{-4}$, for a homogeneous isotropic elastic material, such that

$$\boldsymbol{u}(x,y;t) = \left(\cos\left(\frac{4\pi x}{c_p}\right), \cos\left(\frac{4\pi x}{c_s}\right)\right)\cos(4\pi t), \qquad c_p = \sqrt{\frac{\lambda + 2\mu}{\rho_e}}, \; c_s = \sqrt{\frac{\mu}{\rho_e}}$$

$$\varphi(x,y;t) = \sin(4\pi x)\sin(4\pi t),$$

# Numerical example II

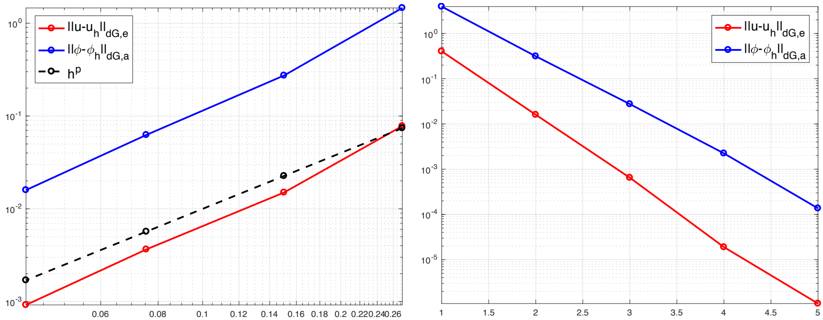
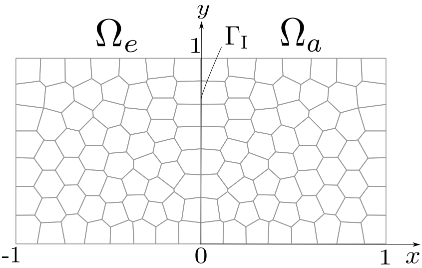
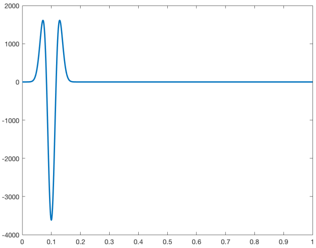



Figure: $\|\boldsymbol{u} - \boldsymbol{u}_h\|_{\mathrm{dG},e}$ and $\|\varphi - \varphi_h\|_{\mathrm{dG},a}$ vs. $h$ (left) and $p$ (right) at $T = 0.8$

# Physical example




$$t \mapsto -2\pi a \left(1 - 2\pi a (t - t_0)^2\right) e^{-\pi a (t-t_0)^2}$$

## Physical example

We simulate a seismic source in the acoustic domain by a point-wise source:

$$f_a(\boldsymbol{x}, t) = -2\pi a \left(1 - 2\pi a(t - t_0)^2\right) e^{-\pi a (t-t_0)^2} \delta(\boldsymbol{x} - \boldsymbol{x}_0), \quad \boldsymbol{x}_0 \in \Omega_a, \ t_0 \in (0, T],$$

$$\boldsymbol{x}_0 = (0.2, 0.5), \quad t_0 = 0.1$$

# Physical example

$t \mapsto \|\boldsymbol{u}(x, y; t)\|_2$ and $t \mapsto |\varphi(x, y; t)|$

# Conclusions & perspectives

## Conclusions

- We proved that the elasto-acoustic problem is well-posed in the continuous setting
- We proved and numerically validated $hp$-convergence results for a discontinuous Galerkin method on polyhedral meshes
- We used the method to simulate an example of physical interest

# Conclusions & perspectives

## Conclusions

- We proved that the elasto-acoustic problem is well-posed in the continuous setting
- We proved and numerically validated $hp$-convergence results for a discontinuous Galerkin method on polyhedral meshes
- We used the method to simulate an example of physical interest

## Perspectives

- Carrying out numerical computations in a 3D setting, using the code `SPEED` (`http://speed.mox.polimi.it/`)
- Considering the case of totally absorbing boundary conditions
- Deducing error estimates for the fully discrete problem
- Consider the more general case of a viscoelastic material response:

$$\boldsymbol{\sigma}(\boldsymbol{u}(\boldsymbol{x},t);t) = \mathbb{C}(\boldsymbol{x},0)\boldsymbol{\varepsilon}(\boldsymbol{u}(\boldsymbol{x},t)) - \int_0^t \frac{\partial \mathbb{C}}{\partial s}(\boldsymbol{x},t-s)\boldsymbol{\varepsilon}(\boldsymbol{u}(\boldsymbol{x},s))\,\mathrm{d}s$$

# References I


P. F. ANTONIETTI, F. BONALDI, AND I. MAZZIERI,
*A high-order discontinuous Galerkin approach to the elasto-acoustic problem*,
Preprint arXiv:1803.01351 [math.NA], submitted, 2018.

A. CANGIANI, Z. DONG, E. H. GEORGOULIS, AND P. HOUSTON,
$hp$-*Version discontinuous Galerkin methods on polygonal and polyhedral meshes*,
SpringerBriefs in Mathematics, Springer International Publishing, 2017.

P. F. ANTONIETTI, P. HOUSTON, X. HU, M. SARTI, AND M. VERANI,
*Multigrid algorithms for hp-version interior penalty discontinuous Galerkin methods on polygonal and polyhedral meshes*,
Calcolo, 54 (2017), pp. 1169–1198.

S. MÖNKÖLA,
*On the accuracy and efficiency of transient spectral element models for seismic wave problems*,
Adv. Math. Phys., (2016).

J. D. DE BASABE AND M. K. SEN,
*A comparison of finite-difference and spectral-element methods for elastic wave propagation in media with a fluid-solid interface*,
Geophysical Journal International, 200 (2015), pp. 278–298.

# References II


H. Barucq, R. Djellouli, and E. Estecahandy,
*Characterization of the Fréchet derivative of the elasto-acoustic field with respect to Lipschitz domains*, J. Inverse Ill-Posed Probl., 22 (2014), pp. 1–8.

H. Barucq, R. Djellouli, and E. Estecahandy,
*Efficient dG-like formulation equipped with curved boundary edges for solving elasto-acoustic scattering problems*, Int. J. Numer. Meth. Engng, 98 (2014), pp. 747–780.

V. Péron,
*Equivalent boundary conditions for an elasto-acoustic problem set in a domain with a thin layer*,
ESAIM Math. Model. Numer. Anal., 48 (2014), pp. 1431–1449.

B. Flemisch, M. Kaltenbacher, and B. I. Wohlmuth,
*Elasto–acoustic and acoustic–acoustic coupling on non-matching grids*,
Int. J. Numer. Meth. Engng, 67 (2006), pp. 1791–1810.

D. Komatitsch, C. Barnes, and J. Tromp,
*Wave propagation near a fluid-solid interface: a spectral-element approach*,
Geophysics, 65 (2000), pp. 623–631.


# Merci de votre attention

MODELLISTICA E CALCOLO SCIENTIFICO
MOX
MODELING AND SCIENTIFIC COMPUTING

SPEED

# Application of Hille–Yosida

Let $\boldsymbol{w} = \dot{\boldsymbol{u}}$, $\phi = \dot{\varphi}$, and $\mathcal{U} = (\boldsymbol{u}, \boldsymbol{w}, \varphi, \phi)$. We introduce

$$\mathbb{H} = \boldsymbol{H}^1_D(\Omega_e) \times \boldsymbol{L}^2(\Omega_e) \times H^1_D(\Omega_a) \times L^2(\Omega_a),$$

with scalar product

$$\begin{aligned}(\mathcal{U}_1, \mathcal{U}_2)_{\mathbb{H}} = (\rho_e \zeta^2 \boldsymbol{u}_1, \boldsymbol{u}_2)_{\Omega_e} + (\mathbb{C}\boldsymbol{\varepsilon}(\boldsymbol{u}_1), \boldsymbol{\varepsilon}(\boldsymbol{u}_2))_{\Omega_e} \\ + (\rho_e \boldsymbol{w}_1, \boldsymbol{w}_2)_{\Omega_e} + (\rho_a \boldsymbol{\nabla}\varphi_1, \boldsymbol{\nabla}\varphi_2)_{\Omega_a} + (c^{-2}\rho_a \phi_1, \phi_2)_{\Omega_a}.\end{aligned}$$

# Application of Hille–Yosida

Let $\boldsymbol{w} = \dot{\boldsymbol{u}}$, $\phi = \dot{\varphi}$, and $\mathcal{U} = (\boldsymbol{u}, \boldsymbol{w}, \varphi, \phi)$. We introduce

$$\mathbb{H} = \boldsymbol{H}^1_D(\Omega_e) \times \boldsymbol{L}^2(\Omega_e) \times H^1_D(\Omega_a) \times L^2(\Omega_a),$$

with scalar product

$$\begin{aligned}(\mathcal{U}_1, \mathcal{U}_2)_{\mathbb{H}} = (\rho_e \zeta^2 \boldsymbol{u}_1, \boldsymbol{u}_2)_{\Omega_e} + (\mathbb{C}\boldsymbol{\varepsilon}(\boldsymbol{u}_1), \boldsymbol{\varepsilon}(\boldsymbol{u}_2))_{\Omega_e} \\ + (\rho_e \boldsymbol{w}_1, \boldsymbol{w}_2)_{\Omega_e} + (\rho_a \boldsymbol{\nabla}\varphi_1, \boldsymbol{\nabla}\varphi_2)_{\Omega_a} + (c^{-2}\rho_a \phi_1, \phi_2)_{\Omega_a}.\end{aligned}$$

Then, we define the operator $A\colon D(A) \subset \mathbb{H} \to \mathbb{H}$ by

$$A\mathcal{U} = \left(-\boldsymbol{w},\ 2\zeta\boldsymbol{w} + \zeta^2\boldsymbol{u} - \rho_e^{-1}\,\mathbf{div}\,\mathbb{C}\boldsymbol{\varepsilon}(\boldsymbol{u}),\ -\phi,\ -c^2\triangle\varphi\right) \quad \forall \mathcal{U} \in D(A),$$

$$\begin{aligned}D(A) = \Big\{\mathcal{U} \in \mathbb{H} : \boldsymbol{u} \in \boldsymbol{H}^{\triangle}_{\mathbb{C}}(\Omega_e),\ \boldsymbol{w} \in \boldsymbol{H}^1_D(\Omega_e),\ \varphi \in H^{\triangle}(\Omega_a),\ \phi \in H^1_D(\Omega_a); \\ (\mathbb{C}\boldsymbol{\varepsilon}(\boldsymbol{u}) + \rho_a\phi\boldsymbol{I})\,\boldsymbol{n}_e = \boldsymbol{0} \text{ on } \Gamma_{\mathrm{I}},\ \ (\boldsymbol{\nabla}\varphi + \boldsymbol{w})\cdot\boldsymbol{n}_a = 0 \text{ on } \Gamma_{\mathrm{I}}\Big\}.\end{aligned}$$

# Application of Hille–Yosida

Let $\boldsymbol{w} = \dot{\boldsymbol{u}}$, $\phi = \dot{\varphi}$, and $\mathcal{U} = (\boldsymbol{u}, \boldsymbol{w}, \varphi, \phi)$. We introduce

$$\mathbb{H} = \boldsymbol{H}^1_D(\Omega_e) \times \boldsymbol{L}^2(\Omega_e) \times H^1_D(\Omega_a) \times L^2(\Omega_a),$$

with scalar product

$$\begin{aligned}(\mathcal{U}_1, \mathcal{U}_2)_{\mathbb{H}} &= (\rho_e \zeta^2 \boldsymbol{u}_1, \boldsymbol{u}_2)_{\Omega_e} + (\mathbb{C}\boldsymbol{\varepsilon}(\boldsymbol{u}_1), \boldsymbol{\varepsilon}(\boldsymbol{u}_2))_{\Omega_e} \\ &\quad + (\rho_e \boldsymbol{w}_1, \boldsymbol{w}_2)_{\Omega_e} + (\rho_a \boldsymbol{\nabla}\varphi_1, \boldsymbol{\nabla}\varphi_2)_{\Omega_a} + (c^{-2}\rho_a \phi_1, \phi_2)_{\Omega_a}.\end{aligned}$$

Then, we define the operator $A\colon D(A) \subset \mathbb{H} \to \mathbb{H}$ by

$$A\mathcal{U} = \left(-\boldsymbol{w},\ 2\zeta\boldsymbol{w} + \zeta^2\boldsymbol{u} - \rho_e^{-1}\,\mathbf{div}\,\mathbb{C}\boldsymbol{\varepsilon}(\boldsymbol{u}),\ -\phi,\ -c^2\triangle\varphi\right) \quad \forall \mathcal{U} \in D(A),$$

$$\begin{aligned}D(A) = \Big\{\mathcal{U} \in \mathbb{H} : \boldsymbol{u} \in \boldsymbol{H}^{\triangle}_{\mathbb{C}}(\Omega_e),\ \boldsymbol{w} \in \boldsymbol{H}^1_D(\Omega_e),\ \varphi \in H^{\triangle}(\Omega_a),\ \phi \in H^1_D(\Omega_a); \\ (\mathbb{C}\boldsymbol{\varepsilon}(\boldsymbol{u}) + \rho_a\phi\boldsymbol{I})\,\boldsymbol{n}_e = \boldsymbol{0} \text{ on } \Gamma_{\mathrm{I}},\ \ (\boldsymbol{\nabla}\varphi + \boldsymbol{w})\cdot\boldsymbol{n}_a = 0 \text{ on } \Gamma_{\mathrm{I}}\Big\}.\end{aligned}$$

Finally, let $\mathcal{F} = (\boldsymbol{0}, \rho_e^{-1}\boldsymbol{f}_e, 0, c^2 f_a)$.

For $\mathcal{F} \in C^1([0,T];\mathbb{H})$ and $\mathcal{U}_0 \in D(A)$,
find $\mathcal{U} \in C^1([0,T];\mathbb{H}) \cap C^0([0,T];D(A))$ s.t.

$$\frac{\mathrm{d}\mathcal{U}}{\mathrm{d}t}(t) + A\mathcal{U}(t) = \mathcal{F}(t), \quad t \in (0,T],$$

$$\mathcal{U}(0) = \mathcal{U}_0.$$