Stabilité des états fondamentaux et des solutions normalisées pour une équation de Schrödinger d'ordre quatre

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- Mass subcritical case
- Mass supercritical case

Introduction

We are interested in the fourth order Schrödinger equation :

$$i\partial_t \psi - \gamma \Delta^2 \psi + \Delta \psi + |\psi|^{2\sigma}\psi = 0, \text{ in } \mathbb{R} \times \mathbb{R}^N \quad \text{(Mixed 4NLS)}$$

where $\sigma, \ \gamma > 0$.
We focus essentially on standing wave solutions namely solutions of the form

$$\psi(t,x) = e^{i\alpha t}u(x), \text{ for some } \alpha \in \mathbb{R}.$$

This ansatz yields to the fourth-order semilinear elliptic equation

$$\gamma \Delta^2 u - \Delta u + \alpha u = |u|^{2\sigma}u, \text{ in } \mathbb{R}^N. \quad \text{(4NLS)}$$

Introduction

Some motivations.

A small fourth order dispersion has been introduced by Karpman and Shagalov to regularize and stabilize solutions to the classical nonlinear Schrödinger equation.

V.I. Karpman and A.G. Shagalov. Stability of solitons described by nonlinear Schrödinger-type equations with higher-order dispersion.
Phys. D, **144**(1-2):194–210, 2000.

Introduction

First, let us consider the Schrödinger equation in arbitrary dimension with a general pure power nonlinearity

$$i\partial_t\psi + \Delta\psi + |\psi|^{2\sigma}\psi = 0,\ \psi(x,0) = \psi_0(x),\ (t,x) \in \mathbb{R}\times\mathbb{R}^N,\ \text{(NLS)}$$

where σ is a given positive real number. We have that

- all solutions to (NLS) exist globally in time and standing waves are orbitally stable if $\sigma N < 2$,
- finite time blow-up may appear and the standing wave solutions become unstable if $\sigma N \geq 2$.

Introduction

For (Mixed 4NLS), Fibich, Ilan and Papanicolaou showed using a combination of stability analysis and numerical simulations that

- all solutions to (Mixed 4NLS) exist globally in time (see also Ben Artzi-Koch-Saut) and standing wave solutions are stable when $\sigma N < 4$ (provided γ is small if $2 \leq N\sigma < 4$),
- standing wave solutions are instable when $\sigma N \geq 4$.

Two observations :

- The case $\sigma = 1$ and $N = 2$ is now subcritical.
- Solutions blowing-up in finite time when $\sigma N \geq 4$ were only recently proved to exist.

Existence of standing wave solutions for 4NLS

They are two natural ways to look for standing wave solutions to (4NLS) :

- minimization under a L^2 mass constraint.
- minimization under a $L^{2\sigma+2}$ constraint.

- The L^2 mass constraint is very natural with respect to the time dependent equation (Mixed 4NLS) (the mass is a conserved quantity).
- The $L^{2\sigma+2}$ constraint is very natural from an elliptic PDE point of view.

The $L^{2\sigma+2}$ constraint which was studied by Bonheure and Nascimento.

Introduction

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Minimizing under a $L^{2\sigma+2}$ constraint

We consider the following minimization problem

$$m = \inf_{u \in M} J_{\gamma,\alpha}(u), \qquad \text{(Min } L^{2\sigma+2} \text{ fixed)}$$

where

$$M = \{u \in H^2(\mathbb{R}^N) : \int_{\mathbb{R}^N} |u|^{2\sigma+2} dx = 1\},$$

and $J_{\gamma,\alpha}$ is the quadratic form defined by

$$J_{\gamma,\alpha}(u) = \gamma \int_{\mathbb{R}^N} |\Delta u|^2 \, dx + \int_{\mathbb{R}^N} |\nabla u|^2 \, dx + \alpha \int_{\mathbb{R}^N} |u|^2 \, dx.$$

Minimizing under a $L^{2\sigma+2}$ constraint

Two observations :

- When $\alpha, \gamma > 0$, $J_{\gamma,\alpha}$ is the square of a norm on H^2.
- If m is achieved by some $u \in M$, then u weakly solves

$$\gamma\Delta^2 u - \Delta u + \alpha u = m|u|^{2\sigma}u.$$

Then $v = m^{\frac{1}{2\sigma}}u$ solves

$$\gamma\Delta^2 v - \Delta v + \alpha v = |v|^{2\sigma}v.$$

Minimizing under a $L^{2\sigma+2}$ constraint

Theorem (Bonheure, Nascimento)

Assume $\alpha, \gamma > 0$ *and* $\sigma N < 4^* = 4N/(N-4)^+$. *Then problem* (Min $L^{2\sigma+2}$ fixed) *has a nontrivial solution. Moreover, when* $\gamma\alpha \leq 1/4$, *this solution has a sign and is radially symmetric.*

Minimizing under a $L^{2\sigma+2}$ constraint

When $\alpha \leq 1/4$ (to simplify we take $\gamma = 1$), (4NLS) can be rewritten as a cooperative system

$$\begin{cases} -\Delta u + \frac{u}{2} & = v, \\ -\Delta v + \frac{v}{2} & = |u|^{2\sigma}u + (\frac{1}{4} - \alpha)u. \end{cases}$$

If we prove that u and v don't change sign then a general result of Busca and Sirakov implies that u and v are strictly radially decreasing.

Minimizing under a $L^{2\sigma+2}$ constraint

Let u be a solution to (Min $L^{2\sigma+2}$ fixed). Define $w \in H^2$ through

$$-\Delta w + \frac{w}{2} = \left| -\Delta u + \frac{u}{2} \right|.$$

Assume by contradiction that $-\Delta u + \dfrac{u}{2}$ changes sign. From the strong maximum principle, we have $w > |u|$. Then, we have

$$J_{1,\alpha}\left(\frac{w}{\|w\|_{L^{2\sigma+2}}}\right) = \frac{\int_{\mathbb{R}^N} (-\Delta w + w/2)^2\,dx - (1/4-\alpha)\int_{\mathbb{R}^N} w^2\,dx}{\|w\|^2_{L^{2\sigma+2}}}$$

$$< \frac{\int_{\mathbb{R}^N} (-\Delta u + u/2)^2\,dx - (1/4-\alpha)\int_{\mathbb{R}^N} u^2\,dx}{\|u\|^2_{L^{2\sigma+2}}}$$

which contradicts the minimality of u.

Introduction

Existence of ground-state solutions: minimizing under a $L^{2\sigma+2}$ constraint

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Stability properties of stationnary solutions to 4NLS

Minimizing under a L^2 mass constraint

Next, we consider the problem with fixed L^2 constraint :

$$I_\gamma(\mu) = \inf_{u \in S_\mu} E_\gamma(u) \qquad \text{(Min } L^2 \text{ fixed)}$$

where

$$S_\mu = \{u \in H^2(\mathbb{R}^N) : \int_{\mathbb{R}^N} |u|^2 \, dx = \mu\}$$

and

$$E_\gamma(u) = \frac{\gamma}{2} \int_{\mathbb{R}^N} |\Delta u|^2 \, dx + \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 \, dx - \frac{1}{2\sigma + 2} \int_{\mathbb{R}^N} |u|^{2\sigma+2} \, dx.$$

Minimizing under a L^2 mass constraint

If u achieves $I_\gamma(\mu)$, then u is a solution to

$$\gamma\Delta^2 u - \Delta u + \alpha u = |u|^{2\sigma} u,$$

where α is the Lagrange multiplier

$$\alpha = \frac{1}{\mu}\left(\int_{\mathbb{R}^N} |u|^{2\sigma+2}\,dx - \gamma\int_{\mathbb{R}^N} |\Delta u|^2\,dx - \int_{\mathbb{R}^N} |\nabla u|^2\,dx\right).$$

Remark : not possible to "scale out" α.

Minimizing under a L^2 mass constraint: mass subcritical case

Theorem (Bonheure, C, Dos Santos, Nascimento)

Assume $\gamma > 0$. If $0 < \sigma N < 2$, then $I_\gamma(\mu)$ is achieved for every $\mu > 0$. If $2 \leq \sigma N < 4$, then there exist a critical mass $\mu_c(\gamma, \sigma)$ such that

(i) *$I_\gamma(\mu)$ is not achieved if $\mu < \mu_c$;*

(ii) *$I_\gamma(\mu)$ is achieved if $\mu > \mu_c$ and $\sigma N = 2$;*

(iii) *$I_\gamma(\mu)$ is achieved if $\mu \geq \mu_c$ and $\sigma N \neq 2$;*

$$\lim_{\gamma \to 0} \mu_c(\gamma, \sigma) = 0.$$

Minimizing under a L^2 mass constraint: mass subcritical case

Sketch of the proof : thanks to Lions' concentration -compactness principle, one can prove that if $I_\gamma(\mu) < 0$, then sequences of minimizers to (Min L^2 fixed) are pre-compact. The main problem to prove that $I_\gamma(\mu) < 0$ is the presence of three terms in E_γ.

Minimizing under a L^2 mass constraint: mass subcritical case

Let $u_\lambda(x) = \lambda^{\frac{N}{2}} u(\lambda x)$. Then,

$$E_\gamma(u_\lambda) = \frac{\gamma \lambda^4}{2} \int_{\mathbb{R}^N} |\Delta u|^2 \, dx + \frac{\lambda^2}{2} \int_{\mathbb{R}^N} |\nabla u|^2 \, dx - \frac{\lambda^{\sigma N}}{2\sigma + 2} \int_{\mathbb{R}^N} |u|^{2\sigma+2}.$$

Difficult to conclude when $2 \leq \sigma N < 4$.
Main idea : use a 3 terms Gagliardo-Nirenberg interpolation inequality.

Minimizing under a L^2 mass constraint: mass critical case

Theorem (Bonheure, C, Gou, Jeanjean)

Let $\sigma N = 4$. There exists a $\mu_N^* > 0$ such that

$$I_\gamma(\mu) := \inf_{u \in S_\mu} E_\gamma(u) = \begin{cases} 0, & 0 < \mu \leq \mu_N^*, \\ -\infty, & \mu > \mu_N^*, \end{cases}$$

For $\mu \in (0, \mu_N^*)$, (Min L^2 fixed) *has no solution and in particular $I_\gamma(\mu)$ is not achieved.*

Minimizing under a L^2 mass constraint: mass supercritical case

In order to find renormalized solutions when the energy is not bounded from below, we consider the following modified minimization problem

$$K(\mu) = \inf_{u \in \mathcal{M}(\mu)} E_\gamma(u) \qquad \text{(Min } L^2 \text{ fixed sup)}$$

where

$$\mathcal{M}(\mu) = \{u \in H^2(\mathbb{R}^N) | \|u\|_{L^2}^2 = \mu \; and \; Q(u) = 0\},$$

and

$$Q(u) = \gamma \int_{\mathbb{R}^N} |\Delta u|^2 \, dx + \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 \, dx - \frac{\sigma N}{2\sigma + 2} \int_{\mathbb{R}^N} |u|^{2\sigma+2} \, dx.$$

Minimizing under a L^2 mass constraint: mass supercritical case

Lemma (Bonheure, C, Gou, Jeanjean)

Let $4 < \sigma N < 4^*$, then E restricted to $\mathcal{M}(\mu)$ is coercive and bounded from below by a positive constant.

Lemma (Bonheure, C, Gou, Jeanjean)

Let $4 < \sigma N < 4^*$. There exists a Palais-Smale sequence $\{u_n\} \subset \mathcal{M}(\mu)$ for E restricted to $S(\mu)$ at the level $K(\mu)$.

Minimizing under a L^2 mass constraint: mass supercritical case

Theorem (Bonheure, C, Gou, Jeanjean)

Let $4 < \sigma N < 4^*$. *Then there exists* $\mu_{N,\sigma} > 0$ *such that for any* $\mu \in (0, \mu_{\sigma,N})$, (Min L^2 fixed sup) *has a solution* u *satisfying* $E_\gamma(u) = K(\mu)$. *Moreover*

$$\mu_{1,\sigma} = \mu_{2,\sigma} = \infty,$$

and

$$\mu_{3,\sigma} = \infty \text{ if } 4 < 3\sigma \leq 6 = 2^* = 2N/(N-2).$$

Minimizing under a L^2 mass constraint: mass supercritical case

Sketch of the proof :

Lemma (Bonheure, C, Gou, Jeanjean)

Let $\{u_n\} \subset \mathcal{M}(\mu)$ be a Palais-Smale sequence for E restricted to $\mathcal{M}(\mu)$. Then there exist $u_\mu \in H^2(\mathbb{R}^N)$ and a sequence $\{\alpha_n\} \subset \mathbb{R}$ such that, up to translation and up to passing to a subsequence,

(i) $u_n \rightharpoonup u_\mu \neq 0$ *in* $H^2(\mathbb{R}^N)$ *as* $n \to \infty$;

(ii) $\alpha_n \to \alpha_\mu$ *in* $\mathbb{R}$ *as* $n \to \infty$;

(iii) $\gamma \Delta^2 u_n - \Delta u_n + \alpha_n u_n - |u_n|^{2\sigma} u_n \to 0$ *in* $H^{-2}(\mathbb{R}^N)$ *as* $n \to \infty$;

(iv) $\gamma \Delta^2 u_\mu - \Delta u_\mu + \alpha_\mu u_\mu = |u_\mu|^{2\sigma} u_\mu$.

In addition, if $\alpha_\mu > 0$ *then* $\|u_n - u_\mu\|_{H^2} \to 0$ *as* $n \to \infty$.

Minimizing under a L^2 mass constraint: mass supercritical case

Lemma (Bonheure, C, Gou, Jeanjean)

If u_μ is a solution to

$$\gamma\Delta^2 u - \Delta u + \alpha_\mu u = |u|^{2\sigma} u \tag{4NLS}$$

with $\|u_\mu\|_2^2 = \mu > 0$, there exists $\mu_{N,\sigma} > 0$ such that $\alpha_\mu > 0$ for any $\mu \in (0, \mu_{N,\sigma})$. Moreover

$$\mu_{1,\sigma} = \mu_{2,\sigma} = \infty,$$

and

$$\mu_{3,\sigma} = \infty \text{ if } 4 < 3\sigma \leq 6 = 2^*.$$

Orbital stability: mass supercritical case

Definition

We say that a solution $u \in H^2(\mathbb{R}^N)$ to (Mixed 4NLS) *is unstable by blow-up in finite time if, for all $\varepsilon > 0$, there exists $v \in H^2(\mathbb{R}^N)$ such that $\|v - u\|_{H^2} < \varepsilon$ and the solution $\phi(t)$ to* (Mixed 4NLS) *with initial data $\phi(0) = v$ blows up in finite time in the H^2 norm.*

Orbital stability: mass supercritical case

Theorem (Bonheure, C, Gou, Jeanjean)

Let $4 \leq \sigma N < 4^*$, $N \geq 2$ and $\sigma \leq 4$. The standing waves associated to radial ground states and renormalized solutions are unstable by blow-up in finite time.

Orbital stability

Definition

Let $\theta(x,t) = e^{i\alpha t}U(x)$ be a standing wave of (Mixed 4NLS). *We say that θ is orbitally stable in* H^2 *if, given $\varepsilon > 0$, there exists $\delta > 0$ such that if $\Phi_0 \in \mathbb{H}^2$ satisfies $\|\Phi_0 - \mathcal{U}\|_{\mathbb{H}^2} < \delta$, then the solution $\Phi(t)$ of* (Mixed 4NLS) *with initial data Φ_0 exists for all $t \geq 0$ and satisfies*

$$d(\Phi(t), \Omega_{\mathcal{U}}) < \varepsilon, \text{ for all } t \geq 0,$$

where

$$\begin{aligned}\Omega_{\mathcal{U}} &:= \{T_1(\theta)T_2(r)\mathcal{U} : \theta, r \in \mathbb{R}\} \\ &\approx \left\{ \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} U(\cdot - r) \\ 0 \end{pmatrix} : \theta, r \in \mathbb{R} \right\},\end{aligned}$$

and

$$d(f,g) := \inf\left\{ \|f - T_1(\theta)T_2(r)g\|_{\mathbb{H}^2} : \theta, r \in \mathbb{R} \right\}.$$

Orbital stability: mass subcritical case

Theorem (Bonheure, C, Dos Santos, Nascimento)

Let $0 < \sigma N < 4$. Suppose that u is a minimizer for (Min L^2 fixed)*.*
Then the standing wave $\psi(t,x) = \exp(i\alpha t)u(x)$ is orbitally stable.

Idea of the proof : same strategy as Cazenave-Lions.

Orbital stability: mass subcritical case

Theorem (Bonheure, C, Dos Santos, Nascimento)

Let $0 < \sigma N < 4$*. Suppose that* u *is a non-degenerate minimizer of* (Min $L^{2\sigma+2}$ fixed) *i.e. if* v *satisfies*

$$\gamma\Delta^2 v - \Delta v + \alpha v = (2\sigma+1)|u|^{2\sigma} v,$$

then there exists $\xi \in \mathbb{R}^N$ *such that* $v(x) = \xi \cdot \nabla u(x)$*, and that the following condition holds*

$$\text{if } v \in H^2(\mathbb{R}^N) \text{ satisfies } \gamma\Delta^2 v - \Delta v + \alpha v = u, \text{ then } \int_{\mathbb{R}^N} uv < 0.$$

Then the standing wave $\psi(t,x) = \exp(i\alpha t)u(x)$ *is orbitally stable.*

Idea of the proof : construction of a Lyapunov function.

Orbital stability: mass subcritical case

Theorem (Bonheure, C, Dos Santos, Nascimento)

Let u be a minimizer of (Min $L^{2\sigma+2}$ fixed). *Then, u is non-degenerate if*

- $N = 1$ *and* $\gamma\alpha \leq 1/4$,
- $N\sigma < 2^*$, $\alpha > 0$ *and* $\gamma > 0$ *small enough.*

Orbital stability: mass supercritical case

Theorem (Bonheure, C, Gou, Jeanjean)

Let $4 \leq \sigma N < 4^*$, $N \geq 2$ and $\sigma \leq 4$. The standing waves associated to radial ground states and renormalized solutions are unstable by blow-up in finite time.

Orbital stability: mass supercritical case

Main ingredient : Boulenger and Lenzmann introduced a localized virial to (Mixed 4NLS). Let $\varphi : \mathbb{R}^N \to \mathbb{R}$ be a radial function such that $D^j\varphi \in L^\infty(\mathbb{R}^N)$, $1 \leq j \leq 6$,

$$\varphi(r) := \begin{cases} \frac{r^2}{2} & \text{for } r \leq 1 \\ const. & \text{for } r \geq 10 \end{cases}, \text{and } \varphi''(r) \leq 1, \text{ for } r \geq 0.$$

Let $R > 0$, we set $\varphi_R(r) := R^2\varphi(\frac{r}{R})$. For $u \in H^2(\mathbb{R}^N)$, we defined the localized virial M_{φ_R} by

$$M_{\varphi_R}[u] := 2\mathrm{Im} \int_{\mathbb{R}^N} \bar{u}\nabla\varphi_R \nabla u \, dx.$$

Orbital stability: mass supercritical case

Lemma (Boulenger, Lenzmann)

Let $\sigma N < 4^*, N \geq 2$, and $R > 0$. Suppose that $u(t) \in C([0,T); H^2_{rad}(\mathbb{R}^N))$ is a solution to (Mixed 4NLS). *Then for any $t \in [0,T)$,*

$$\begin{aligned}\frac{d}{dt}M_{\varphi_R}[u(t)] &\leq 4N\sigma E_\gamma(u(t)) - (2N\sigma - 8)\gamma\|\Delta u(t)\|_2^2 - (2N\sigma - 4)\|\nabla u(t)\|_2^2 \\ &\quad + O\left(\frac{1}{R^4} + \frac{\|\nabla u(t)\|_2^2}{R^2} + \frac{\|\nabla u(t)\|_2^\sigma}{R^{\sigma(N-1)}} + \frac{\mu}{R^2}\right) \\ &= 8Q(u(t)) + O\left(\frac{1}{R^4} + \frac{\|\nabla u(t)\|_2^2}{R^2} + \frac{\|\nabla u(t)\|_2^\sigma}{R^{\sigma(N-1)}} + \frac{\mu}{R^2}\right).\end{aligned}$$

Moreover, if $\sigma \leq 2$, we have

$$\frac{d}{dt}M_{\varphi_R}[u(t)] \leq 8Q(u(t)) + 0\left(\frac{1}{\eta R^2} + \eta^{1/2}\right), \qquad (1)$$

for, $R \geq 1$ and $0 < \eta < 1$.

Orbital stability: mass supercritical case

Sketch of the proof : Let $u \in H^2_{rad}(\mathbb{R}^N)$ be a ground state solution. We can prove that there exists $\lambda > 1$ sufficiently close to 1 such that $v(r) = \lambda^{N/4} u(\sqrt{\lambda} r)$ satisfies

$$\tilde{E}_\alpha(v) < \tilde{E}_\alpha(u), \ Q(v) < 0, \ I(v) < 0 \textit{ and } \|u - v\|_{H^2} \le \varepsilon,$$

where

$$\tilde{E}_\alpha(u) = \frac{\gamma}{2}\|\Delta u\|_2^2 + \frac{1}{2}\|\nabla u\|_2^2 + \frac{\alpha}{2}\|u\|_2^2 - \frac{1}{2\sigma+2}\|u\|_{2\sigma+2}^{2\sigma+2},$$

$$I(u) = \gamma\|\Delta u\|_2^2 + \|\nabla u\|_2^2 + \alpha\|u\|_2^2 - \|u\|_{2\sigma+2}^{2\sigma+2},$$

$$Q(u) = \gamma\|\Delta u\|_2^2 + \frac{1}{2}\|\nabla u\|_2^2 - \frac{\sigma N}{2(2\sigma+2)}\|u\|_{2\sigma+2}^{2\sigma+2}.$$

Let $\phi(t) \in C([0, T); H^2_{rad}(\mathbb{R}^N))$ be the unique solution to (Mixed 4NLS) with initial datum $\phi(0) = v$, where $T > 0$ is the maximum existence time to $\phi(t)$.

Orbital stability: mass supercritical case

First step: There exists a constant $a > 0$ (not depending on t) such that, for all $t \in [0, T)$,

$$\tilde{E}_\alpha(\phi(t)) < \tilde{E}_\alpha(u),\ Q(\phi(t)) < -a,\ \textit{and}\ I(\phi(t)) < 0.$$

Second step: There exist a constant $\delta > 0$ such that

$$\frac{d}{dt} M_{\varphi_R}[\phi(t)] \leq -\delta \|\nabla\phi(t)\|_2^2 \text{ for } t \in [0, T), \qquad \text{(Virial)}$$

and a $t_1 \geq 0$ such that

$$M_{\varphi_R}[\phi(t)] < 0 \text{ for } t \geq t_1.$$

Idea : consider two cases depending on the sign of

$$||\nabla\phi(t)||_2^2 - \frac{4N\sigma E(v)}{\mu(N\sigma - 2)}.$$

Orbital stability: mass supercritical case

Final step: Suppose by contradiction that $T = \infty$, then integrating (Virial) on $[t_1, t]$, we have that

$$M_{\varphi_R}[\phi(t)] \leq -\delta \int_{t_1}^{t} \|\nabla \phi(s)\|_2^2 ds.$$

Now using Cauchy-Schwarz's inequality, we get

$$|M_{\varphi_R}[\phi(t)]| \leq 2\|\nabla \varphi_R\|_\infty \|\phi(t)\|_2 \|\nabla \phi(t)\|_2 \leq C\|\nabla \phi(t)\|_2.$$

Thus for some $\tau > 0$,

$$M_{\varphi_R}[\phi(t)] \leq -\tau \int_{t_1}^{t} |M_{\varphi_R}[\phi(s)]|^2 ds.$$

Setting $z(t) := \int_{t_1}^{t} |M_{\varphi_R}[\phi(s)]|^2 ds$, this rewrites as

$$z'(t) \geq \tau^2 z(t)^2.$$

Thank you for your attention.