On singularity formation for Prandtl's equations and Burgers equation with transverse viscosity

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Main issue

Goal : construct precise blow-up solutions to

$$(vB) \quad \partial_t u + u\partial_x u - \partial_{yy} u = 0, \quad (x,y) \in \mathbb{R}, \quad u : [0,T) \times \mathbb{R}^2 \to \mathbb{R}.$$

Motivations :

- Simple toy model, everything can be computed explicitely.
- How does an additional effect affects a blow-up dynamics it does not prevent ?
- Mixed hyperbolic/parabolic problem.
- Anisotropic singularities are still poorly understood, see also [C.-Merle-Raphaël, preprint 2017] and [M-R-Szeftel, preprint 2017] for the semi-linear heat equation in high dimensions.
- Same goal for Prandtl's equations.

Relation with blow-up for Prandtl's equations

Prandtl's equations on the upper half plane $(x,y) \in \mathbb{R} \times [0,+\infty)$ with vanishing flow at infinity :

$$(Prandtl) \quad \begin{cases} \partial_t u + u\partial_x u + v\partial_y u - \partial_{yy} u = 0, \quad u_x + v_y = 0, \\ u(t,x,0) = 0 = v(t,x,0) \end{cases}$$

admit solutions becoming singular in finite time, but the precise structures of the singularities are unknown.

Namely, if u solves $(Prandtl)$ is odd in x, the trace $\xi(t,y) := -u_x(t,0,y)$ solves on the vertical ray $y \in [0,+\infty)$:

$$(*) \quad \xi_t - \xi^2 + \left(\int_0^y \xi(t,\tilde{y}) d\tilde{y} \right) \xi_y - \xi_{yy} = 0, \quad \xi(0) = 0.$$

[E-Engquist CPAM 1997] showed that some solutions to $(*)$ blow up in finite time, see also [Kukavica-Vicol-Wang Adv. Math. 2017, Galaktionov-Vazquez Adv. Differential Equ. 1999]. Seminal numerical results by [Van-Dommelen-Shen J. Comput. Phys. 1980].

Relation with blow-up for Prandtl's equations

Theorem [Galaktionov-Vazquez Adv. Differential Equ. 1999, C.-Ghoul-Ibrahim-Masmoudi, work under progress]

There exists a stable blow-up behaviour for $(*)$ with solutions such that as $t \to T$:

$$\xi(t,y) = \frac{1}{\lambda(t)} Q\left((y - y(t))\mu(t)\sqrt{\lambda(t)}\right) + \varepsilon(t,y).$$

where, for some $\eta > 0$,

$$Q(Z) = \cos^2\left(\frac{Z}{2}\right) \mathbf{1}\left\{-\pi \leq Z \leq \pi\right\},$$

$$\lambda(t) \sim (T-t), \quad \mu \to \mu^* > 0, \quad y(t) \sim \frac{\pi}{\sqrt{T-t}}, \quad \|\varepsilon\|_{L^\infty} \lesssim (T-t)^{-1+\eta}$$

However, the solution to $(Prandtl)$ might blow up before the time T associated to the reduced equation. What happens outside the vertical axis $\{x = 0\}$? What is the role of the reduced equation, i.e. of the infinitesimal behaviour near the vertical axis ?

We solved this problem for the simplified model (vB).

Self-similarity in shocks for the inviscid Burgers equation

(vB) for solutions independent on the transversal variable $u(t,x,y)=U(t,x)$ reduces to the inviscid Burgers equation

$$(Burgers)\quad U_t+UU_x=0,\quad U(0,x)=U_0(x)$$

for which some solutions become singular in finite time. We now explain **the role of self-similarity in this singularity** formation.

There holds the formula using the characteristics

$$U(t,x)=U_0(\Phi_t^{-1}(x)),\quad \Phi_t(\tilde{x})=\tilde{x}+tU_0(\tilde{x})$$

and $\partial_x U\to+\infty$ becomes singular at time $T=(-\min(\partial_x U_0))^{-1}$ (shock formation).

Invariances : If U is a solution to $(Burgers)$ then so is

$$\frac{\mu}{\lambda}U\left(\frac{t-t_0}{\lambda},\frac{x-x_0-ct}{\mu}\right)+c.$$

Wlog if U blows up then we assume

$$(HP)\quad T=0,\quad U_x(-1,x) \text{ minimal at } 0 \text{ with } U_x(-1,0)=-1 \text{ and } U(-1,0)=0.$$

Self-similarity refers to such solutions with a non-trivial stabilizer

$$\mathcal{G}_s:=\left\{(\lambda,\mu)\in(0,+\infty)^2,\frac{\mu}{\lambda}U\left(\frac{t}{\lambda},\frac{x}{\mu}\right)=U(t,x)\right\}\neq\{(1,1)\}.$$

Self-similarity in shocks for Burgers equation

Proposition [Invariant blow-up solutions]

$U \in \mathcal{C}^1$ solves (*Burgers*), satisfies (*HP*), $U \neq x/t$, and $\mathcal{G}_s$ is non-trivial if and only if one the following holds.

(i) Self-similarity (SS) : There exists $i > 0$ with

$$U(t,x) = \mu^{-1}(-t)^{\frac{1}{2i}} \Psi_i \left(\mu \frac{x}{(-t)^{1+\frac{1}{2i}}} \right), \ \mu > 0,$$

where Ψ_i is a profile which is analytic if $i \in \mathbb{N}^*$, else C^{1+2i}.

(ii) Discrete self-similarity (DSS) : There exists $i > 0$, $\lambda > 1$ such that

$$U(t,x) = \lambda^{\frac{k}{2i}} U \left(\frac{t}{\lambda^k}, \frac{x}{\lambda^{k(1+\frac{1}{2i})}} \right), \quad \forall k \in \mathbb{Z}$$

and $U \notin C^{1+2i}$.

Self-similarity in shocks for Burgers equation

Proposition [The family $(\Psi_i)_{i\in\mathbb{N}^*}$]

$\Psi_i : \mathbb{R} \to \mathbb{R}$ is odd, analytic and solves the stationary self-similar equation :

$$-\frac{1}{2i}\Psi_i + \left(1+\frac{1}{2i}\right)X\partial_X\Psi_i + \Psi_i\partial_X\Psi_i = 0,$$

and the implicit equation :

$$X = -\Psi_i - \Psi_i^{2i+1},$$

with behaviour :

$$\Psi_i(X) \underset{X\to 0}{=} -X + X^{2i+1} + ..., \qquad \Psi_i(X) \underset{X\to-\infty}{\sim} |X|^{\frac{1}{2i+1}},$$

and in particular the fundamental one is :

$$\Psi_1(X) = \left(-\frac{X}{2} + \left(\frac{1}{27}+\frac{X^2}{4}\right)^{\frac{1}{2}}\right)^{\frac{1}{3}} + \left(-\frac{X}{2} - \left(\frac{1}{27}+\frac{X^2}{4}\right)^{\frac{1}{2}}\right)^{\frac{1}{3}}.$$

They are the attractors of non-degenerate smooth singular solutions. Dynamics of perturbations can be studied explicitly.

Behaviour on the vertical axis

Assume u solves (vB) is **odd in x and even in y** and such that

$$\partial_x^j u_0(0,y)=0 \quad \text{for } j=2,...,2i$$

and define

$$\xi(t,y):=-\partial_x u(t,0,y), \quad \zeta(t,y)=\partial_x^{2i+1}u(t,0,y).$$

Then u remains odd in x and even in y, with

$$\partial_x^j u_0(0,y)=0 \quad \text{for } j=2,...,2i,$$

and

$$\begin{cases} (1) & \xi_t-\xi^2-\xi_{yy}=0, \\ (2) & \zeta_t-(2i+2)\xi\zeta-\zeta_{yy}=0. \end{cases}$$

(1) is the 1-dimensional semi-linear heat equation. (2) is a linearly forced heat equation.

Behaviour on the vertical axis

Theorem [Giga-Kohn CPAM 1985, Herrero-Velazquez Ann. Sc. Norm. Super. Pisa Cl. Sci. 1992, Bricmont-Kupiainen Nonlinearity 1994, Merle-Zaag Duke Math. J. 1997]

There exists an open set of solutions to

$$\xi_t - \xi^2 - \xi_{yy} = 0$$

that blow up as $t \to T$ with

$$\xi = \frac{1}{T-t} \frac{1}{1+\frac{y^2}{8(T-t)|\log(T-t)|}} + O_{L^\infty(\mathbb{R})}\left(\frac{1}{(T-t)|\log(T-t)|^\eta}\right), \quad \eta > 0, \text{ as } t \to T,$$

and instable solutions for each $k \in \mathbb{N}$, $k \geq 2$ such that

$$\xi = \frac{1}{T-t} \frac{1}{1+\frac{ay^{2k}}{(T-t)}} + O_{L^\infty(\mathbb{R})}\left(\frac{\left((T-t)^{\frac{1}{2k}} + |y|\right)^{\frac{1}{2}}}{T-t+y^{2k}}\right), \quad a > 0, \text{ as } (t,y) \to (T,0).$$

(optimal bound in the second case by [C.-Ghoul-Masmoudi, preprint 2018]).

Main result

Main Theorem 1 [C.-G.-M., preprint 2018]

For any $i \in \mathbb{N}^*$ and $\mu > 0$, there exists an open set of solutions to (vB) that are smooth and compactly supported, odd in x and even in y, blowing up in finite time T with

$$u = \frac{(T-t)^{\frac{1}{2i}}}{\mu}\left(1+\frac{y^2}{8(T-t)|\log(T-t)|}\right)^{\frac{1}{2i}}\Psi_i\left[\frac{1}{\left(1+\frac{y^2}{8(T-t)|\log(T-t)|}\right)^{1+\frac{1}{2i}}}\frac{\mu x}{(T-t)^{1+\frac{1}{2i}}}\right]$$
$$+(T-t)^{\frac{1}{2i}}\tilde{u}(X,Z)$$

where $X = x/(T-t)^{1+1/(2i)}$, $Z = y/\sqrt{(T-t)|\log(T-t)|}$ and

$$\tilde{u} \to 0 \quad \text{in } C^1 \text{ on compact sets}$$

and

$$\left\|\partial_x\left((T-t)^{\frac{1}{2i}}\tilde{u}(X,Z)\right)\right\| \lesssim (T-t)^{-1}|\log(T-t)|^{-\eta}, \quad \eta > 0.$$

Main result

Main Theorem 2 [C.-G.-M., preprint 2018]

For any $i, k \in \mathbb{N}^*$, $k \geq 2$, and $a, \mu > 0$, there exist solutions to (vB) that are smooth and compactly supported, odd in x and even in y, blowing up in finite time T with

$$u = \frac{(T-t)^{\frac{1}{2i}}}{\mu}\left(1+\frac{ay^{2k}}{(T-t)}\right)^{\frac{1}{2i}}\Psi_i\left[\frac{1}{\left(1+\frac{ay^{2k}}{(T-t)}\right)^{1+\frac{1}{2i}}}\frac{\mu x}{(T-t)^{1+\frac{1}{2i}}}\right] + (T-t)^{\frac{1}{2i}}\tilde{u}(X,Z)$$

where $X = x/(T-t)^{1+1/(2i)}$, $Z = y/(T-t)^{1/(2k)}$ and

$$\tilde{u} \to 0 \quad \text{in } C^1 \text{ on compact sets}$$

and

$$\left\| \partial_x \left((T-t)^{\frac{1}{2i}} \tilde{u}(X,Z) \right) \right\| \lesssim (T-t)^{-1+\eta}, \quad \eta > 0.$$

Proof, Renormalisation

For simplicity :

$$i=1, \quad k=2, \quad a=1, \quad \mu=1,$$

$$\Rightarrow \quad \text{profile } (T-t)^{\frac{1}{2}}\left(1+\frac{y^4}{(T-t)}\right)^{\frac{1}{2}}\Psi_1\left[\frac{1}{\left(1+\frac{y^4}{(T-t)}\right)^{\frac{3}{2}}}\frac{x}{(T-t)^{\frac{3}{2}}}\right].$$

We zoom at the blow-up using the self-similar variables :

$$X=\frac{x}{(T-t)^{\frac{3}{2}}}, \quad Y=\frac{y}{(T-t)^{\frac{1}{2}}}, \quad Z=\frac{y}{(T-t)^{\frac{1}{4}}}, \quad s=-\log(T-t)$$

$$v(s,X,Y)=(T-t)^{-\frac{1}{2}}u(t,x,y), \quad f(s,Y)=-\partial_X v(s,0,Y), \quad g(s,Y)=\partial_X^3 v(s,0,Y).$$

Then v solves

$$(vB') \quad \partial_s v-\frac{1}{2}v+\frac{3}{2}X\partial_X v+\frac{1}{2}Y\partial_Y+v\partial_X v-\partial_{YY}v=0.$$

Proof, Precise behaviour on the vertical axis

Proposition

Given $0 < T \ll 1$ small enough (equivalently $s_0 \gg 1$ large enough), there exists a solution (f, g) to

$$\begin{cases} (1') & \partial_s f + f + \frac{Y}{2}\partial_Y f - f^2 - \partial_{YY} f = 0, \\ (2') & \partial_s g + 4g + \frac{Y}{2}\partial_Y g - 4fg - \partial_{YY} g = 0, \end{cases}$$

such that

$$f = \frac{1}{1+Z^4} + \tilde{f}, \quad g = \frac{6}{(1+Z^4)^4} + \tilde{g},$$

where (which propagates for derivatives)

$$|\tilde{f}| \lesssim \frac{e^{-\frac{1}{16}s}(1+|Z|)^{\frac{1}{2}}}{(1+|Z|)^4}, \quad |\tilde{g}| \lesssim \frac{e^{-\frac{1}{16}s}(1+|Z|)^{\frac{1}{2}}}{(1+|Z|)^{16}}.$$

Proof, Leading order profile

Once the behaviour of $\partial_X v$ and $\partial_X^3 v$ on the vertical axis $\{X = 0\}$ is known, how to extend the solution in 2d ? In Z variables the equation for v reads

$$\partial_s v - \frac{1}{2} v + \frac{3}{2} X \partial_X v + \frac{1}{4} Z \partial_Z v + v \partial_X v - e^{-\frac{s}{2}} \partial_{ZZ} v = 0.$$

with asymptotic stationary equation

$$-\frac{1}{2} v + \frac{3}{2} X \partial_X v + \frac{1}{4} Z \partial_Z v + v \partial_X v = 0.$$

Proof, Leading order profile

Proposition

The asymptotic stationary equation admits the solution

$$\Theta = (1+Z^4)^{\frac{1}{2}} \Psi_1 \left(\frac{X}{(1+Z^4)^{\frac{3}{2}}} \right)$$

and the linearised operator

$$\mathcal{L} = -\frac{1}{2} + \frac{3}{2} X\partial_X + \frac{1}{4} Z\partial_Z + \partial_X \Theta + \Theta \partial_X$$

admits the eigenvalues

$$\mathcal{L}\psi_{j,\ell} = \left(\frac{j}{2} + \frac{\ell}{4} - \frac{3}{2} \right) \psi_{j,\ell}, \quad \psi_{j,\ell} = Z^\ell (1+Z^4)^{\frac{j}{2}-1} \frac{\Psi_1^j \left(\frac{X}{(1+Z^4)^{\frac{3}{2}}} \right)}{1 + 3\Psi_1^2 \left(\frac{X}{(1+Z^4)^{\frac{3}{2}}} \right)}$$

and the maximum principle-type estimate

$$\left\| \frac{e^{-\mathcal{L}s}\varepsilon_0}{\psi_{j,\ell}} \right\|_{L^\infty(\mathbb{R}^2)} \leq e^{-\left(\frac{j}{2} + \frac{\ell}{4} - \frac{3}{2} \right) s} \left\| \frac{\varepsilon_0}{\psi_{j,\ell}} \right\|_{L^\infty(\mathbb{R}^2)} .$$

Proof, Approximate blow-up profile

We take as approximate blow-up profile for $0 < \delta \ll 1$:

$$Q = \chi_{|Y| \le \delta e^{\frac{s}{2}}} \tilde{\Theta} + (1 - \chi_{|Y| \le \delta e^{\frac{s}{2}}}) \Theta_e$$

where for the functions f and g previously studied :

$$\tilde{\Theta} = \mu^{-1}(s,Y) f^{-\frac{1}{2}}(s,Y) \Psi_1 \left(\mu(s,Y) f^{\frac{3}{2}}(s,Y) X \right), \quad \mu(s,Y) = \left(\frac{g(s,Y)}{6 f^4(s,Y)} \right)^{\frac{1}{2}},$$

$$\Theta_e = \left(-f(s,Y) X + \frac{g(s,Y)}{6} X^3 \right) e^{-\left(\frac{X}{(1+Z^6)}^4 \right)}.$$

We take the Ansatz for the solution

$$v = Q + \varepsilon, \quad \text{then} \quad \partial_X^j \varepsilon(s,0,Y) = 0 \quad \text{for } j = 0,1,2,3,4.$$

Proof, Bootstrap

Let the vector field

$$A = \left(\frac{3}{2} X + (1+Z^4)^{\frac{3}{2}} \Psi_1 \left(\frac{X}{(1+Z^4)^{\frac{3}{2}}} \right) \right) \partial_X \qquad (A \sim X\partial_X).$$

Proposition

One can bootstrap the estimates for $0 < \kappa \ll 1$, $q \gg 1$, integers $0 \leq j_1 + j_2 \leq 2$:

$$\left(\int_{\mathbb{R}^2} \frac{((\partial_Z^{j_1} A^{j_2} \varepsilon)^{2q}}{\psi_{4,0}^{2q}(X,Z)} \frac{dXdY}{|X|\langle Y \rangle} \right)^{\frac{1}{2q}} \lesssim e^{-(\frac{1}{2}-\kappa)s},$$

and for $0 \leq j_1 + j_2 \leq 2$ and $j_2 \geq 1$:

$$\left(\int_{\mathbb{R}^2} \frac{(((Y\partial_Y^{j_1}) A^{j_2} \varepsilon)^{2q}}{\psi_{4,0}^{2q}(X,Z)} \frac{dXdY}{|X|\langle Y \rangle} \right)^{\frac{1}{2q}} \lesssim e^{-(\frac{1}{2}-\kappa)s}.$$

Note that $\psi_{4,0} \sim X^4$ as $X \to 0$ hence the need of the cancellation $\partial_X^j \varepsilon = 0$ for $j = 0,1,2,3,4$. This yields pointwise estimate thanks to the Sobolev embedding

$$\|u\|_{L^\infty(\mathbb{R}^2)}^{2q} \leq C(q) \left(\int_{\mathbb{R}^2} u^{2q} \frac{dXdY}{|X|\langle Y \rangle} + \int_{\mathbb{R}^2} (X\partial_X u)^{2q} \frac{dXdY}{|X|\langle Y \rangle} + \int_{\mathbb{R}^2} (\langle Y \rangle \partial_Y u)^{2q} \frac{dXdY}{|X|\langle Y \rangle} \right)$$

Proof, Bootstrap

Main ingredients of the bootstrap analysis :

- The approximate profile Q generates an error which vanishes on the axis $\{X=0\}$ up to order 4. Anisotropic weighted estimates are possible thanks to the analysis performed previously on the vertical axis.
- To take derivatives : A commutes with the leading order transport operator $\partial_s+\mathcal{L}$. ∂_Z and $Y\partial_Y$ commute up to well-localised remainders involving A.
- There holds the following linear estimate if u and R solve

$$u_s-\frac{1}{2}u+\frac{3}{2}X\partial_X u+\frac{1}{2}Y\partial_Y u+\partial_X\Theta u+\Theta\partial_X u-\partial_{YY}u=R,$$

$$\begin{aligned}
&\frac{d}{ds}\left(\frac{1}{2q}\int_{\mathbb{R}^2}\frac{u^{2q}}{\psi_{j,0}^{2q}(X,Z)}\frac{dXdY}{|X|\langle Y\rangle}\right)\\
\leq\ & -\left(\frac{j-3}{2}-\frac{C}{q}\right)\int_{\mathbb{R}^2}\frac{u^{2q}}{\psi_{j,0}^{2q}(X,Z)}\frac{dXdY}{|X|\langle Y\rangle}-\frac{2q-1}{q^2}\int\frac{|\partial_Y(u^q)|^2}{\psi_{j,0}^{2q}(X,Z)}\frac{dXdY}{|X|\langle Y\rangle}\\
&+\int\frac{u^{2q-1}R}{\psi_{j,0}^{2q}(X,Z)}\frac{dXdY}{|X|\langle Y\rangle}.
\end{aligned}$$

Conclusion

- Smooth enough non-degenerate shock formations of Burgers equation $U_t + UU_x = 0$ are self-similar, and everything can be computed easily explicitely.
- Some singularities of Burgers with transversal viscosity $u_t + uu_x - u_{yy} = 0$ are given by a backward self-similar solution of Burgers equation along the x variable, whose two scaling parameters depend on the transversal variable y, and evolve according to a parabolic system including the well-known semi-linear heat equation. The leading order blow-up profile is explicit and anisotropic.
- The remaining part of the solution having enough cancellations on the vertical axis is damped and sent to infinity in renormalised variables in an anisotropic way.
- Model for connexions between self-similar blow-ups behaviours.
- Hope for description of singularities of Prandtl's equations, for the moment precise information for reduced equations are obtained.

Thank you for your attention ! !