The incompressible Navier-Stokes equations in vacuum

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Incompressible Navier-Stokes equations in an open subset $\Omega \subset \mathbb{R}^d$:

$$(NS): \begin{cases} u_t + \operatorname{div}(u \otimes u) - \mu\Delta u + \nabla P = 0 & \text{in } \mathbb{R}_+ \times \Omega \\ \operatorname{div} u = 0 & \text{in } \mathbb{R}_+ \times \Omega. \end{cases}$$

- Energy balance : $\displaystyle \frac{1}{2}\|u(t)\|_2^2 + \mu \int_0^t \|\nabla u\|_2^2 \, d\tau = \frac{1}{2}\|u_0\|_2^2.$
- Leray solutions : any divergence free $u_0 \in L^2(\Omega)$ generates at least one global weak solution satisfying the energy inequality.
- $d = 2$: Those solutions are unique.
- $d = 3$: Those solutions are unique in a suitable class of *smoother* solutions.

The **inhomogeneous** **incompressible Navier-Stokes equations** read:

$$(INS): \begin{cases} (\rho u)_t + \operatorname{div}(\rho u \otimes u) - \mu\Delta u + \nabla P = 0 & \text{in } \mathbb{R}_+ \times \Omega \\ \operatorname{div} u = 0 & \text{in } \mathbb{R}_+ \times \Omega \\ \rho_t + \operatorname{div}(\rho u) = 0 & \text{in } \mathbb{R}_+ \times \Omega. \end{cases}$$

- Energy balance : $\frac{1}{2}\|\sqrt{\rho(t)}\,u(t)\|_2^2 + \mu\int_0^t \|\nabla u\|_2^2\,d\tau = \frac{1}{2}\|\sqrt{\rho_0}\,u_0\|_2^2.$
- Conservation of L^p norms of the density and of $\inf \rho(t)$.
- Global weak solutions with finite energy for any pair (ρ_0, u_0) such that $\rho_0 \in L^\infty(\Omega)$ with $\rho_0 \geq 0$, and $\sqrt{\rho_0}u_0 \in L^2(\Omega)$ with $\operatorname{div} u_0 = 0$.
- Even if $d = 2$, uniqueness in the class of finite energy solutions is a widely open question.

Is (INS) a good model for mixture of non-reacting fluids ?

1. Can we solve uniquely (INS) if ρ_0 is *discontinuous* across an interface ?
2. Is the solution unique for such a ρ_0 ?
3. Can we allow for vacuum regions ?
4. Is the regularity of interfaces preserved during the evolution ?

• Having (at least) $\nabla u \in L^1(0,T;L^\infty(\Omega))$ is fundamental both for preserving the "density patch" structure and for proving the uniqueness.

• Even for $d=2$ and for the heat equation, having just $u_0 \in L^2(\Omega)$ does not ensure $\nabla u \in L^1(0,T;L^\infty(\Omega))$.

The main statement

Theorem (Global existence and uniqueness in the 2D torus)

Consider any data (ρ_0, u_0) *in* $L^\infty(\mathbb{T}^2) \times H^1(\mathbb{T}^2)$ *with* $\rho_0 \geq 0$ *and* $\operatorname{div} u_0 = 0$. *Then System* (INS) *supplemented with data* (ρ_0, u_0) *admits a* **unique global solution** $(\rho, u, \nabla P)$ *that satisfies the energy equality, the conservation of total mass and momentum,*

$$\rho \in L^\infty(\mathbb{R}_+; L^\infty), \quad u \in L^\infty(\mathbb{R}_+; H^1), \quad \sqrt{\rho} u_t, \nabla^2 u, \nabla P \in L^2(\mathbb{R}_+; L^2)$$

and also, for all $1 \leq r < 2$, $1 \leq m < \infty$ *and* $T \geq 0$,

$$\nabla(\sqrt{t} P),\ \nabla^2(\sqrt{t} u) \in L^\infty(0, T; L^r) \cap L^2(0, T; L^m).$$

Furthermore, we have $\sqrt{\rho} u \in \mathcal{C}(\mathbb{R}_+; L^2)$ *and* $\rho \in \mathcal{C}(\mathbb{R}_+; L^p)$ *for all* $p < \infty$.

- Similar statement in 2D bounded domains, if taking $u_0 \in H^1_0(\Omega)$.
- Similar statement in the 3D case, under some smallness condition on u_0.

Definition (of a weak solution)

The pair (ρ, u) is a weak solution of (INS) if for all $t \in [0,T)$ and function ϕ in $\mathcal{C}_c^\infty([0,T) \times \Omega; \mathbb{R})$, we have

$$\langle \rho(t), \phi(t)\rangle - \langle \rho_0, \phi_0\rangle - \int_0^t \langle \rho, \partial_t \phi\rangle \, d\tau - \int_0^t \langle \rho u, \nabla\phi\rangle \, d\tau = 0, \tag{1}$$

$$\int_0^t \langle u, \nabla \phi\rangle \, d\tau = 0, \tag{2}$$

and for all divergence-free function $\varphi \in \mathcal{C}_c^\infty([0,T) \times \Omega; \mathbb{R}^d)$,

$$\langle \rho(t) u(t), \varphi(t)\rangle - \langle \rho_0 u_0, \varphi_0\rangle - \int_0^t \langle \rho u, \partial_t \varphi\rangle \, d\tau - \int_0^t \langle \rho u \otimes u, \nabla\varphi\rangle \, d\tau + \mu \int_0^t \langle \nabla u, \nabla \varphi\rangle \, d\tau = 0 \tag{3}$$

where $\langle \cdot, \cdot \rangle$ designates the distribution bracket in Ω.

Main steps of the proof:

1. Global-in-time estimates for Sobolev regularity of u.
2. Sobolev regularity of u_t and time weights.
3. Shift of regularity and integrability : from time to space variable.
4. The existence scheme.
5. *Lagrangian coordinates* and uniqueness.

Assume with no loss of generality that

$$\int_{\mathbb{T}^2} \rho_0 \, dx = \mu = 1 \quad \text{and} \quad \int_{\mathbb{T}^2} \rho_0 u_0 \, dx = 0.$$

Step 1: global-in-time Sobolev estimates

- Take the L^2 scalar product of $\rho(u_t + u\cdot\nabla u) - \Delta u + \nabla P = 0$ with u_t:

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}^2}|\nabla u|^2dx + \int_{\mathbb{T}^2}\rho|u_t|^2\,dx \leq \frac{1}{2}\int_{\mathbb{T}^2}\rho|u_t|^2\,dx + \frac{1}{2}\int_{\mathbb{T}^2}\rho|u\cdot\nabla u|^2\,dx.$$

From $-\Delta u + \nabla P = -(\rho u_t + \rho u\cdot\nabla u)$ and $\operatorname{div}\Delta u = 0$, we have

$$\|\nabla^2 u\|_{L^2}^2 + \|\nabla P\|_{L^2}^2 = \|\rho(\partial_t u + u\cdot\nabla u)\|_{L^2}^2 \leq 2\rho^*\left(\int_{\mathbb{T}^2}\rho|u_t|^2\,dx + \int_{\mathbb{T}^2}\rho|u\cdot\nabla u|^2\,dx\right).$$

Hence

$$\frac{d}{dt}\int_{\mathbb{T}^2}|\nabla u|^2dx + \frac{1}{2}\int_{\mathbb{T}^2}\rho|u_t|^2\,dx + \frac{1}{4\rho^*}\left(\|\nabla^2 u\|_{L^2}^2 + \|\nabla P\|_{L^2}^2\right) \leq \frac{3}{2}\int_{\mathbb{T}^2}\rho|u\cdot\nabla u|^2\,dx.$$

- Apply Hölder and Gagliardo-Nirenberg inequality, and use $\rho \leq \rho^* := \|\rho_0\|_{L^\infty}$:

$$\int_{\mathbb{T}^2}\rho|u\cdot\nabla u|^2\,dx \leq \rho^*\|u\|_{L^4}^2\|\nabla u\|_{L^4}^2 \leq C\rho^*\|u\|_{L^2}\|\nabla u\|_{L^2}^2\|\nabla^2 u\|_{L^2}$$
$$\leq \frac{1}{12\rho^*}\|\nabla^2 u\|_{L^2}^2 + C(\rho^*)^3\|u\|_{L^2}^2\|\nabla u\|_{L^2}^2\|\nabla u\|_{L^2}^2.$$

- If $\rho \geq \rho_* > 0$, then $\|u\|_{L^2}^2 \leq \rho_*^{-1}\|\sqrt{\rho}u\|_{L^2}^2$. Combine the basic energy inequality with Gronwall lemma.

Step 1: global-in-time Sobolev estimates (continued)

Lemma (B. Desjardins, 1997)

If $\int_{\mathbb{T}^2} \rho\, dx = 1$ *and* $\int_{\mathbb{T}^2} \rho z\, dx = 0$ *then*

$$\left(\int_{\mathbb{T}^2} \rho z^4\, dx\right)^{\frac{1}{2}} \leq C\|\sqrt{\rho} z\|_{L^2}\|\nabla z\|_{L^2} \log^{\frac{1}{2}}\left(e + \|\rho - 1\|_{L^2}^2 + \frac{\rho^*\|\nabla z\|_{L^2}^2}{\|\sqrt{\rho} z\|_{L^2}^2}\right). \quad (4)$$

- Write $\displaystyle\int_{\mathbb{T}^2} \rho|u\cdot\nabla u|^2 dx \leq \sqrt{\rho^*}\left(\int_{\mathbb{T}^2} \rho|u|^4\, dx\right)^{\frac{1}{2}}\|\nabla u\|_{L^4}^2$

and use (4) with $z = u$, energy balance and $ab \leq a^2/2 + b^2/2$:

$$\int_{\mathbb{T}^2} \rho|u\cdot\nabla u|^2 dx \leq \frac{1}{12\rho^*}\|\nabla^2 u\|_{L^2}^2$$
$$+C(\rho^*)^2\|\sqrt{\rho_0}\, u_0\|_{L^2}^2\|\nabla u\|_{L^2}^2\|\nabla u\|_{L^2}^2 \log\left(e + \|\rho_0 - 1\|_{L^2}^2 + \rho^*\frac{\|\nabla u\|_{L^2}^2}{\|\sqrt{\rho_0}\, u_0\|_{L^2}^2}\right).$$

We eventually get

$$\frac{d}{dt}X \leq f X \log(e+X),$$

with $f(t) := C_0\|\nabla u(t)\|_{L^2}^2$ for some suitable $C_0 = C(\rho_0, u_0)$ and

$$X(t) := \int_{\mathbb{T}^2} |\nabla u(t)|^2\,dx + \frac{1}{2}\int_{\mathbb{T}^2}\Big(\rho|u_t|^2 + \frac{1}{4\rho^*}\big(|\nabla^2 u|^2 + |\nabla P|^2\big)\Big)dx.$$

Hence

$$(e+X(t)) \leq (e+X(0))^{\exp(\int_0^t f(\tau)\,d\tau)} \leq (e+X(0))^{\exp\,(C_0\|\sqrt{\rho_0}u_0\|_{L^2}^2)}.$$

- However $X<\infty$ does not imply $\nabla u \in L^1_{loc}(\mathbb{R}_+; L^\infty(\mathbb{T}^2))$.

Step 2: Sobolev regularity of u_t

Take the L^2 scalar product of $\rho(u_t+u\cdot\nabla u)-\Delta u+\nabla P=0$ with u_{tt}:

$$\frac12\frac{d}{dt}\int_{\mathbb{T}^2}\rho|u_t|^2\,dx+\int_{\mathbb{T}^2}|\nabla u_t|^2\,dx=\int_{\mathbb{T}^2}\big(\rho_t u_t-\rho_t u\cdot\nabla u-\rho u_t\cdot\nabla u\big)\cdot u_t\,dx.$$

- For $(\sqrt{\rho}u_t)|_{t=0}$ to be defined, we need the compatibility condition

$$-\Delta u_0=\sqrt{\rho_0}g+\nabla P_0\quad\text{with}\quad g\in L^2. \tag{5}$$

- Condition (5) *is not* needed if *we compensate the singularity at time* 0 *by some power of* t: take the scalar product of $\rho(u_t+u\cdot\nabla u)-\Delta u+\nabla P=0$ with tu_{tt}:

$$\|\sqrt{\rho t}\,u_t\|_{L^2}+\int_0^t\|\nabla\sqrt{\tau}\,u_t\|_{L^2}^2\,d\tau\leq h(t),$$

where h is a nondecreasing nonnegative function with $h(0)=0$ (use Step 1 and energy identity).

Step 3: Shift of regularity from time to space variable

- Step 1 gives $\nabla u \in L^\infty(\mathbb{R}_+; L^2)$, $\nabla u \in L^2(\mathbb{R}_+; H^1)$, $\nabla P, \sqrt{\rho} u_t \in L^2(\mathbb{R}_+ \times \mathbb{T}^2)$.
- Step 2 gives $\sqrt{\rho t}\, u_t \in L^\infty_{loc}(\mathbb{R}_+; L^2)$ and $\nabla \sqrt{t}\, u_t \in L^2_{loc}(\mathbb{R}_+; L^2)$.
- Step 3: Use Stokes equation:

$$\begin{cases} -\Delta \sqrt{t}\, u + \nabla \sqrt{t}\, P = -\sqrt{t}\, \rho u_t - \sqrt{t}\, \rho u \cdot \nabla u, \\ \operatorname{div} \sqrt{t}\, u = 0. \end{cases}$$

  Steps 1,2 + embedding imply that for all $T > 0$, the right-hand side is almost in $L^2(0, T; L^\infty)$. Hence so do $\nabla^2 \sqrt{t}\, u$ and $\nabla \sqrt{t}\, P$.

- This implies that $\nabla u \in L^1_{loc}(0, T; L^\infty)$.

Step 4: The existence scheme

- Smooth out ρ_0 and make it positive :

$$\rho_0^n \in \mathcal{C}^1(\mathbb{T}^2;]0,\infty[) \quad \text{with} \quad \rho_0^n \rightharpoonup \rho_0 \quad \text{in} \quad L^\infty \text{ weak } \star .$$

- Solve (INS) with data (ρ_0^n, u_0). From prior works (e.g. Ladyzhenskaya & Solonnikov (78) and others) we get a global 'smooth' solution (ρ^n, u^n).

- From the previous steps, we get uniform estimates.

- By interpolation, we have $(u^n)_{n\in\mathbb{N}}$ is bounded in $H^{\frac{1}{8}}([0,T]\times\mathbb{T}^2)$ for all $T>0$.

- Use compact embeddings to get strong convergence for $(u^n)_{n\in\mathbb{N}}$ and pass to the limit.

- $\rho^n \to \rho$ in $\mathcal{C}([0,T];L^2)$ (similar as for weak solutions).

Step 4: Lagrangian coordinates and uniqueness

Lagrangian coordinates:

$$\bar\rho(t,y) := \rho(t,x), \quad \bar u(t,y) := u(t,x) \text{ and } \bar P(t,y) := P(t,x) \text{ with } \boxed{x := X(t,y)}$$

where X is the flow of u defined by

$$\boxed{X(t,y) = y + \int_0^t u(\tau, X(\tau,y))\,d\tau.}$$

Hence $DX(t,y) = \mathrm{Id} + \displaystyle\int_0^t D\bar u(\tau,y)\,d\tau.$

The INS equations in Lagrangian coordinates:

- $\bar\rho$ is time independent.
- $(\bar u, \bar P)$ satisfies

$$\begin{cases} \rho_0 \bar u_t - \mathrm{div}\,(A^T A \nabla \bar u) + {}^T\!A \cdot \nabla \bar P = 0, \\ \mathrm{div}\,(A\bar u) = {}^T\!A : \nabla \bar u = 0, \end{cases}$$

with

$$A = (D_y X)^{-1} = \sum_{k=0}^{+\infty} (-1)^k \left(\int_0^t D\bar u(\tau,\cdot)\,d\tau \right)^k.$$

Step 4: Lagrangian coordinates and uniqueness (continued)

- Consider two solutions (ρ_1, u_1, P_1) and (ρ_2, u_2, P_2) for the same data (ρ_0, u_0).
- Lagrangian coordinates: $(\rho_i, u_i, P_i) \rightsquigarrow (\rho_0, \bar{u}_i, \bar{P}_i), \quad i = 1, 2.$
- $(\delta u, \delta P) := (u_2 - u_1, P_2 - P_1)$ fulfills

$$\begin{cases} \rho_0 \delta u_t - \operatorname{div}(A_1{}^T A_1 \nabla \delta u) + {}^T A_1 \cdot \nabla \delta P = \operatorname{div}\big((A_2{}^T A_2 - A_1{}^T A_1)\nabla \bar{u}_2\big) + {}^T \delta A \cdot \nabla \bar{P}_2, \\ \operatorname{div}(A_1 \delta u) = \operatorname{div}(\delta A\, \bar{u}_2) = {}^T \delta A : \nabla \bar{u}_2. \end{cases}$$

Decompose δu into $\delta u = w + z$, where is a suitable solution to:

$$\operatorname{div}(A_1 w) = \operatorname{div}(\delta A\, \bar{u}_2) = {}^T \delta A : \nabla \bar{u}_2, \qquad w|_{t=0} = 0,$$

and z fulfills $z(0) = 0$ and

$$\begin{cases} \rho_0 z_t - \operatorname{div}(A_1{}^T A_1 \nabla z) + {}^T A_1 \cdot \nabla \delta P \\ \qquad = \operatorname{div}\big((A_2{}^T A_2 - A_1{}^T A_1)\nabla \bar{u}_2\big) + {}^T \delta A \cdot \nabla \bar{P}_2 - \rho_0 w_t + \operatorname{div}(A_1{}^T A_1 \nabla z), \\ \operatorname{div}(A_1 z) = 0. \end{cases}$$

Step 4: Lagrangian coordinates and uniqueness (continued)

- We have $\operatorname{div}(A_1 w) = g$ where $A_1 \approx \mathrm{Id}$, $\det A_1 = 1$ and $g = \operatorname{div} R$.

This may be recast as $\Phi(v) = v$ with

$$\Phi(v) := \nabla\Delta^{-1}\operatorname{div}((\mathrm{Id} - A_1)v + R).$$

After some computation:

$$\int_0^T \|\nabla w\|_{L^2}^2\,dt \leq c(T)\int_0^T \|\nabla\delta u\|_{L^2}^2\,dt \quad \text{with} \quad \lim_{T\to 0} c(T) = 0. \tag{6}$$

- From $\rho_0 z_t - \operatorname{div}({A_1}^T A_1 \nabla z) + {}^T A_1 \cdot \nabla\delta P = \cdots$ and $\operatorname{div}(A_1 z) = 0$, we get

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{T}^2} \rho_0 |z|^2\,dx + \int_{\mathbb{T}^2} |{}^T A_1 \cdot \nabla z|^2 dx = \cdots.$$

After some computation:

$$\sup_{t\in[0,T]} \|\sqrt{\rho_0}\, z(t)\|_{L^2}^2 + \int_0^T \|\nabla z\|_{L^2}^2\,dt \leq c(T)\int_0^T \|\nabla\delta u\|_{L^2}^2\,dt. \tag{7}$$

- Putting (6) and (7) together yields $\nabla\delta u \equiv 0$ whence $w \equiv 0$, $\sqrt{\rho_0}\, z \equiv 0$ and $\nabla z \equiv 0$. Finally $\delta u \equiv 0$.