Comportements asymptotiques de l'équation de Vlasov-Poisson-Fokker-Planck

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Vlasov-Poisson-Fokker-Planck equation

We consider the dimensionless Vlasov-Poisson-Fokker-Planck equation for a plasma

$$\begin{cases} \partial_t f + v \cdot \nabla_x f - \nabla_x \phi \cdot \nabla_v f = \dfrac{1}{\tau} \nabla_v \cdot (v f + \nabla_v f) \\[2ex] -\delta^2 \Delta_x \phi = \rho - \rho_*(x), \qquad \rho(t,x) = \displaystyle\int_{\mathbb{R}^2} f(t,x,v)\,\mathrm{d}v \end{cases}$$

in a 2D periodic box $x \in \mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$, $v \in \mathbb{R}^2$.

Two dimensionless parameters

- The scaled Debye length $\delta > 0$
- The scaled mean free path / Knudsen number $\tau > 0$

Steady state satisfies

$$\begin{cases} v \cdot \nabla_x f_\infty - \nabla_x \phi_\infty^\delta \cdot \nabla_v f_\infty \;=\; \dfrac{1}{\tau}\,\mathrm{div}_v\left(v f_\infty + \nabla_v f_\infty\right)\;, \\ -\delta^2 \Delta_x \phi_\infty^\delta \;=\; \rho_\infty - \rho_*\,, \qquad \rho_\infty \;=\; \displaystyle\int_{\mathbb{R}^2} f_\infty \,\mathrm{d}v. \end{cases}$$

and is given by

$$f_\infty(x,v) = M(v) e^{-\phi_\infty^\delta(x)}, \qquad M(v) = \frac{1}{2\pi} e^{-\frac{1}{2}|v|^2}\,.$$

with $\phi_\infty^\delta(x)$ satisfying the **Poisson-Boltzmann equation**

$$-\delta^2 \Delta_x \phi_\infty^\delta = e^{-\phi_\infty^\delta} - \rho_*\,.$$

Lemma

For any $\rho_* \in H^{-1}(\mathbb{T}^2)$ such that $\int_{\mathbb{T}^2} \rho_* = 1$, the Poisson-Boltzmann equation possesses a unique weak solution $\phi_\infty^\delta \in H^1(\mathbb{T}^2)$ and moreover $\int_{\mathbb{T}^2} e^{-\phi_\infty^\delta} = 1$.

We are interested in two asymptotic regimes

- Strong collisions / Diffusive regime $\tau \ll 1$
- Evanescent collisions / Inviscid regime $\tau \gg 1$

We want to derive uniform in τ and δ estimates in order to

- Quantify long-time behavior (exponential convergence to f_∞) with explicit asymptotic rates in τ.
- Derive asymptotic models on appropriate time scales.

Far-from-exhaustive state of the art : *Bouchut-Dolbeault, Diff. Int. Eq 95, JMPA 99*, *Hwang-Jang, Contin. Dyn. Syst. Ser. B 2013*, *Hérau-Thomann, J. Funct. Anal. 2016*...

Design of a hypocoercive functional

VPFP in perturbative form

- **Perturbative unknowns:** $h = (f - f_\infty)/f_\infty$ and $E = -\nabla_x(\phi - \phi_\infty^\delta)$.

$$\partial_t h + v \cdot \nabla_x h - \nabla_x \phi_\infty^\delta \cdot \nabla_v h - E \cdot v + E \cdot (\nabla_v - v) h + \frac{1}{\tau}(v - \nabla_v) \cdot \nabla_v h = 0$$

- **Hilbertian setting:** $\mathcal{H} = L^2(\mathbb{T}^2 \times \mathbb{R}^2, \mu)$ with $\mathrm{d}\mu = f_\infty(x, v)\mathrm{d}x\mathrm{d}v$ and canonical scalar product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$.
  $\mathcal{H}_0 = \{h \in \mathcal{H}$ such that $h f_\infty$ is mean free$\}$
- **Operators:** Following Villani, Memoirs AMS, 09 we introduce

$$A = \nabla_v\,, \qquad B = v \cdot \nabla_x - \nabla_x \phi_\infty^\delta \cdot \nabla_v\,, \qquad L_\tau = \frac{1}{\tau} A^* A + B$$

where $A^* = v - \nabla_v$ and $B^* = -B$. Then, VPFP rewrites

$$\boxed{\partial_t h + L_\tau h = E \cdot A^*(1 + h)\,, \qquad E = \delta^{-2} \nabla_x \Delta_x^{-1} n}$$

Hypocoercivity on a toy model: Consider $X(t)$ and $V(t)$ solving

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} X \\ V \end{pmatrix} + \underbrace{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}}_{B}\begin{pmatrix} X \\ V \end{pmatrix} + \frac{1}{\tau}\underbrace{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}}_{A^*A}\begin{pmatrix} X \\ V \end{pmatrix} = 0$$

- Set
$$L_\tau = \frac{1}{\tau}A^*A + B$$
By an explicit computation $|||e^{-L_\tau t}||| \leqslant e^{-\kappa(\tau)\,t}$ with $\kappa(\tau) = O_{\tau\to 0}(\tau)$ and $\kappa(\tau) = O_{\tau\to+\infty}(\tau^{-1})$
- Energy estimate in canonical norm
$$\frac{\mathrm{d}}{\mathrm{d}t}(X^2 + V^2) + \frac{2}{\tau}V^2 = 0$$
- Energy estimate in equivalent norm for small enough $\varepsilon(\tau)$
$$\frac{\mathrm{d}}{\mathrm{d}t}(X^2 + V^2 + \varepsilon(\tau)X\,V) + \kappa_X(\tau)\,X^2 + \kappa_X(\tau)\,V^2 \leqslant 0$$

A weighted H^1 hypocoercive functional

$$\begin{aligned}|||h|||_t^2 \; = \; &\|h\|^2 + \gamma_1 \tau^{\beta_1} \min\left(1, \tfrac{t}{\tau}\right) \|\nabla_v h\|^2 \\ &+ \gamma_2 \tau^{\beta_2} \min\left(1, \tfrac{t}{\tau}\right)^3 \|\nabla_x h\|^2 + 2\gamma_3 \tau^{\beta_3} \min\left(1, \tfrac{t}{\tau}\right)^2 \langle \nabla_v h, \nabla_x h \rangle\end{aligned}$$

- **Time weights** quantify hypoelliptic (regularization) effects
    - Technique introduced by Hérau-Nier, ARMA 04
- **Weights in** τ quantify the size of each derivative. Parameters β_1, β_2 and β_3 will be chosen to close estimates uniformly in $\tau \to 0$ or $\tau \to +\infty$.
- **Full Lyapunov functional**

$$\boxed{\mathcal{E}_t(h) = |||h|||_t^2 + \delta^2 \left(1 + \gamma_1 \tau^{\beta_1} \min\left(1, \tfrac{t}{\tau}\right)\right) \|E\|_{L^2}^2}$$

- **We want to establish** (under conditions on parameters)

$$\mathcal{E}_{t_2}(h(t_2)) + \theta \int_{t_1}^{t_2} \mathcal{D}_s(h(s))\mathrm{d}s \leqslant \mathcal{E}_{t_1}(h(t_1))$$

- **Dissipation** associated with the Lyapunov functional is

$$\begin{aligned}\mathcal{D}_t(h) = \tau^{-1}\|\nabla_v h\|^2 + \gamma_1\tau^{\beta_1-1}\min\left(1,\tfrac{t}{\tau}\right)\left(\|\nabla_v h\|^2 + \|\nabla_v^2 h\|^2\right)\\ + \gamma_2\tau^{\beta_2-1}\min\left(1,\tfrac{t}{\tau}\right)^3\|\nabla_v\nabla_x h\|^2 + \gamma_3\tau^{\beta_3}\min\left(1,\tfrac{t}{\tau}\right)^2\|\nabla_x h\|^2\end{aligned}$$

- **Exponential decay** is a consequence of the Poincaré inequality:

$$\|h\|^2 \leqslant K\,\|e^{\phi_\infty^\delta}\|_{L^\infty(\mathbb{T}^2)}\,\|e^{-\phi_\infty^\delta}\|_{L^\infty(\mathbb{T}^2)}\left(\|\nabla_v h\|^2 + \|\nabla_x h\|^2\right).$$

which yields

$$K\,\tau^{\beta_*}\,\mathcal{E}_t(h(t)) \leqslant \mathcal{D}_t(h(t)).$$

for some β_* depending on $\beta_1, \beta_2, \beta_3$

Main results

Let $\rho_* \in W^{1,p}(\mathbb{T}^2)$, with $p > 2$ s.t. $\int_{\mathbb{T}^2} \rho_* = 1$. Choose $\tau_0 > 0$, and a ball radius for the initial data $R_0 > 1$.

Theorem (H., Rodrigues)

There are positive constants θ_0 and K such that

$$\forall(\tau, \delta), \quad \tau \leqslant \tau_0\,, \quad \delta \geqslant K\,(1 + R_0^{1/2})$$

and for any f_0 s.t. $\|f_0\|_{L^2(M^{-1})} \leqslant R_0$ and $\int_{\mathbb{T}^2 \times \mathbb{R}^2} f_0 = 1$, then VPFP has a unique strong solution f starting from f_0, and for any $t \geqslant 0$

$$\boxed{\|f(t,\cdot,\cdot) - f_\infty\|_{L^2(M^{-1})} \;\leqslant\; K\,\|f_0 - f_\infty\|_{L^2(M^{-1})}\, e^{-\theta_0\,\tau\,t}}$$

and if g solves VPFP starting from g_0

$$\|f(t,\cdot,\cdot) - g(t,\cdot,\cdot)\|_{L^2(M^{-1})} \;\leqslant\; K\,\|f_0 - g_0\|_{L^2(M^{-1})}\, e^{-\theta_0\,\tau\,t}\,.$$

Evanescent collisions $\tau \gg 1$

Theorem (H., Rodrigues)

For any $\varepsilon > 0$, there are positive constants θ_0 and K such that

$$\forall(\tau,\delta), \quad \tau \geqslant \tau_0\,, \quad \delta \geqslant K\,(1 + R_0^{1/2})\,\tau^{7/15+\varepsilon}$$

and for any f_0 s.t. $\|f_0\|_{L^2(M^{-1})} \leqslant R_0$ and $\int_{\mathbb{T}^2\times\mathbb{R}^2} f_0 = 1$, then VPFP has a unique strong solution f starting from f_0, and for any $t \geqslant 0$

$$\boxed{\|f(t,\cdot,\cdot) - f_\infty\|_{L^2(M^{-1})} \ \leqslant\ K\,\|f_0 - f_\infty\|_{L^2(M^{-1})}\, e^{-\theta_0\,\frac{t}{\tau}}}$$

and if g solves VPFP starting from g_0

$$\|f(t,\cdot,\cdot) - g(t,\cdot,\cdot)\|_{L^2(M^{-1})} \ \leqslant\ K\,\|f_0 - g_0\|_{L^2(M^{-1})}\, e^{-\theta_0\,\frac{t}{\tau}}\,.$$

On the size of the nonlinearity

- **Strong collisions** $\tau \to 0$: $\delta \geqslant K\,(1 + R_0^{1/2})$
  - Constraint on δ is the same for linearized equation
  - Thanks to uniformity in τ, we can use estimates to study hydrodynamic limits
- **Evanescent collisions** $\tau \to +\infty$: $\delta \geqslant K\,(1 + R_0^{1/2})\,\tau^{7/15+\varepsilon}$
  - Constraint on δ comes from non-linear terms
  - Uniformity in τ would contradict unstability of double-humped equilibria Penrose, Phys. of Fluids,60 ;Guo-Strauss, Ann. IHP, 95
  - Heuristic justification of the power :

$$y'(t) \;=\; \frac{1 + y(t)}{\delta^2} y(t) - \frac{1}{\tau} y(t)$$

If $y(0) = R_0 > 0$ then exponential decay to 0 occurs if and only if $\delta \geqslant \sqrt{1 + R_0}\,\tau^{1/2}$

We build solutions and close non-linear estimates by a Banach fixed-point argument

- We fix $R_0 > 0$ and choose $h_0 \in \mathcal{H}_0$ such that $\|h_0\| \leqslant R_0$.
- We introduce $X_R = \left\{ h \,\middle|\, \mathcal{E}(h) = \sup_{t \geqslant 0} \mathcal{E}_t(h(t,\cdot,\cdot)) < R^2 \right\}$ and $Y_R = \left\{ h \,\middle|\, \mathcal{E}(h) + \theta \int_0^\infty D_t(h(t,\cdot,\cdot))\,\mathrm{d}t < R^2 \right\}$ with $R = 2R_0$.
- Let

$$\Phi : g \mapsto h$$

where h starts from h_0 and solves the linear equation

$$\partial_t h + L_\tau h = E_h \cdot v + E_g \cdot A^* h,$$

with E_g, E_h the electric fields computed from g and h.
- We show that $\Phi(X_R) \subset Y_R \subset X_R$ and that Φ is a contraction map.

$$\mathcal{E}_t(h(t)) \; + \; \int_0^t \mathcal{D}_s(h(s))\,\mathrm{d}s \; \leqslant \; \|h_0\|^2 \, + \, \mathrm{I} \, + \, \mathrm{II}_\delta \, + \, \mathrm{III}_\delta(g) \, + \, \mathrm{IV}_\delta(g)$$

- $\mathrm{I} \; \leqslant \; \theta \displaystyle\int_0^t \mathcal{D}_s(h(s))\,\mathrm{d}s$ with $\theta \in (0,1)$ uniformly for small τ if

$$\max\left(1, \tfrac{\beta_1+\beta_2}{2}\right) \leqslant \beta_3 \leqslant \min\left(2\beta_1 + 1, \beta_2 - 1\right)$$

  Optimal decay rate obtained for $\beta_3 = 1$, that forces $\beta_1 = 0$ and $\beta_2 = 2$.

- $\mathrm{II}_\delta + \mathrm{III}_\delta(g) \; \leqslant \; \dfrac{K(1+R)}{\delta^2} \displaystyle\int_0^t \mathcal{D}_s(h(s))\,\mathrm{d}s$

- $\mathrm{IV}_\delta(g) \; \leqslant \; \dfrac{KR}{\delta^2} \displaystyle\int_0^{\min(t,\tau)} \tau^{1-2\eta} \left(\tfrac{s}{\tau}\right)^{-3\eta} \|h(s)\|^2\,\mathrm{d}s$

  Non-linear growth during short-time regularization
  $\|E\|_{L^\infty(\mathbb{T}^2)} \; \leqslant \; \frac{K_\eta}{\delta^2} \|h\|^{1-\eta} \|\nabla_x h\|^\eta$ singular when $t \to 0$ since $\nabla_x h_0 \notin \mathcal{H}$.

Concluding remarks

Additional results

- Derivation of hydrodynamic limits on appropriate (τ-dependent time scales) when $\tau \to 0$ (*e.g.* strong convergence of macroscopic density to parabolic / diffusion limit)

Multiscale hypocoercive estimates

- Can be adapted to many situations to study **long-time behavior** with asymptotic dependence of rates on scaling parameters and to derive **hydrodynamic limits**
  - For Boltzmann with small Knudsen number Briant, JDE, 15
  - For VPFP with strong magnetic field / small electron mass H., Rodrigues, preprint 17
  - Kinetic models multiple scales and uncertainty Jin, Zhu, preprint 17, Jin, Liu, preprint 17

M. H. and L. M. Rodrigues. Large-Time Behavior of Solutions to Vlasov-Poisson-Fokker-Planck Equations: From Evanescent Collisions to Diffusive Limit. *Journal of Statistical Physics*, Feb 2018.